Shearing flow over a wavy boundary

By T. BROOKE BENJAMIN
Department of Engineering, University of Cambridge

(Received 10 December 1958)

A theoretical study is made of shearing flows bounded by a simple-harmonic wavy surface, the main object being to calculate the normal and tangential stresses on the boundary. The type of flow considered is approximately parallel in the absence of the waves, being exemplified by two-dimensional boundary layers over a plane. Account is taken of viscosity; but, as the Reynolds number is assumed to be large, its effects are seen to be confined within narrow 'friction layers', one of which adjoins the wave and another surrounds the 'critical point' where the velocity of flow equals the wave velocity. The boundary conditions are made as general as possible by including the three cases where respectively the boundary is rigid, flexible yet still solid, or completely mobile as if it were the interface with a second fluid.

The theory is developed on the model of stable laminar flow, although it is proposed that the same theory may usefully be applied also to examples of turbulent flow considered as 'pseudo-laminar' with velocity profiles corresponding to the mean-velocity distribution. Use is made of curvilinear co-ordinates which follow the contour of the wave-train. This admits a linearized form of the problem whose validity requires only that the wave amplitude be small in comparison with the wavelength, even when large velocity gradients exist close to the boundary. The analysis is made largely without restriction to particular forms of the velocity profile; but eventually consideration is given to the example of a linear profile and the example of a boundary-layer profile approximated by a quarter-period sinusoid. In § 7 some general methods are set out for the treatment of disturbed boundary-layer profiles: these apply with greatest precision to thin boundary layers, but are also useful for the initially very steep but on the whole fairly diffuse profiles which occur in most practical instances of turbulent flow over waves.

The phase relationships found between the stresses and the wave elevation are discussed for several examples, and their interest in connexion with problems of wave generation by wind is pointed out. It is shown that in most circumstances the stresses are distributed in much the same way as if the leeward slopes of the waves were sheltered. For instance, the pressure distribution often has a substantial component in phase with the wave slope, just as if a wake were formed behind each wave crest—although of course actual separation effects are outside the scope of the present theory. In this aspect, the analysis amplifies the work of Miles (1957).

1. Introduction

The main purpose of this work is to estimate the stresses on wavy surfaces bounding several kinds of shearing flow at high yet finite Reynolds number, including uniform shearing flow and flows characterized by a boundary layer. To make a tractable theoretical model, the primary motion is taken to be parallel and in the direction of the waves, so the disturbance due to the waves is two-dimensional and strictly periodic in the x co-ordinate. We neglect the spontaneous fluctuations with respect to time which would usually occur in the real case (i.e. due to instability of a laminar flow or to fully developed turbulence); but while the theory applies rigorously only to stable laminar flows, we consider there is sufficient justification to take the bold step of applying it also to examples of turbulent flow, assuming that the mean-velocity distribution is disturbed by the waves in approximately the same way as the velocity distribution in an equivalent laminar flow. This course seems to bring the work into much closer relation to practical problems than if only laminar flows were considered, even though admittedly the neglect of interactions between the turbulence and the wave motion constitutes a very severe simplification for which *a priori* theoretical justification is difficult to find. However, some justification for this, together with a clear statement of what it implies mathematically, was given by Miles (1957, see particularly the Appendix) whose work bears closely on the present problem and will be frequently cited in what follows.

Expressions will be found for the surface stresses rather more readily in the case of a rigid boundary than in the case where flexure of the boundary allows the wave-train to travel in the direction of flow. In the latter case the wave velocity may equal the fluid velocity at a certain distance from the boundary; and this 'critical point' becomes a vital factor in the analysis in much the same way as it does in the stability theory of parallel flows (see Lin 1955, chapters 3, 8). However, the results for the latter case have special interest in that they might be applied to problems of wave generation by flow over a mobile boundary, for instance wind over a water surface. Such problems are commonly approached by considering a simple wave-train of arbitrary wavelength and speed superposed on the equilibrium state and then finding conditions under which the reaction of the disturbed flow is just sufficient to maintain the waves. Knowledge of the forces exerted on the wave surface therefore comprises an intermediate step towards the solution of the problem; and the remaining steps need be concerned only with the matter (e.g. the water) under the action of these forces. Although the subject of wave generation is incidental to the main contents of this paper, some features of the results will be discussed which are particularly interesting from this aspect.

Most previous work relevant to this investigation has in fact been directly concerned with questions of how the action of wind may give rise to waves of one sort and another. Particularly in the vast literature concerning wave generation on deep water,† several distinct theoretical models have been used to account for

† Two other practical problems of wave generation by wind have become important in recent times: the first relates to the flutter of membranes and thin panels, and the second to the instability of liquid films dragged over a solid boundary by a gas stream, the latter

flow over a prescribed wavy boundary; and it is desirable to note what relation some of these have to the present contribution.

The classical Kelvin–Helmholtz theory is based on the simplest model (Lamb 1932, §§ 234, 268). This assumes uniform flow initially in the air and the water, the velocities being discontinuous at the interface. Viscosity is neglected, so that the disturbed motion arising when the interface is perturbed by a simple wave-train can be described by a velocity potential. Apart from the obvious gravity force, the only effect of the air stream upon the water surface is a periodic pressure which, if the wave amplitude does not vary with time, is in opposite phase to the wave elevation and thus acts exactly contrary to the total effect of gravity and surface tension. The pressure is proportional to the square of the air velocity relative to the waves, and is just the same for a rigid wave-train if a uniform primary velocity is assumed for this case also. Such a rudimentary model has obvious physical limitations; but we shall see in § 7 that the pressure component given by the Kelvin–Helmholtz theory is consistent with the results of a more realistic theory valid for thin boundary layers: it will be shown that this pressure component is the only stress remaining in the limit as the Reynolds number is taken uniformly to infinity.

An approach to the problem of water-wave generation similar in principle to the Kelvin–Helmholtz theory, but taking account of viscosity, was used by Wuest (1949) and Lock (1954); and the model on which their analyses were based is of the type considered in this paper. Laminar flow is assumed in the air and the water; and the stability of the motion with respect to wavy disturbances is investigated in the way usual for problems of boundary-layer stability (i.e. the velocity profile is assumed to have negligible variation over distances comparable with the wavelength, so that linearized equations of motion of the Orr–Sommerfeld type are obtained—see Lin (1955, chapter 5)). The velocity profiles considered by Wuest were arbitrarily chosen as rough approximations to actual boundary-layer profiles; but Lock considered an exact profile which he had previously calculated for a boundary layer starting from a certain point upstream. Now, as both the air and water motions can be separately unstable even in the absence of the additional factor presented by the mobility of the interface, stability analyses of this sort are extremely complicated. Thus, although Lock's results are undoubtedly valuable, they are of such complexity that it would seem unlikely they could ever be checked by experimental observation. In this paper we shall study disturbed laminar flows of the kind treated by Lock; but we shall ignore possible instability of the flow and accordingly take the disturbance to be stationary relative to the waves on the boundary, thus enabling us to study much more clearly the interesting effects due specifically to the presence of the waves.

Another model which is to be taken as an example in this paper has previously been used by Feldman (1957), who considered a stability problem akin to Wuest's and Lock's but for a horizontal liquid film in contact with a semi-infinite air

problem having importance in chemical engineering and in connexion with the cooling of rocket motors (Knuth 1954). As in the water-wave problem, the main difficulty in these two also is to account for the disturbed air flow; and so the present work may be said to have equal bearing on all three.

stream. He also assumed parallel laminar flow, but introduced the simplification that both fluids are initially in uniform shearing motion (i.e. plane Couette flow). This rules out the occurrence of 'self-instability' of the air stream as encountered in Lock's work, since semi-infinite Couette flow is completely stable (this was proved by Zondek & Thomas (1953)). But, as Feldman was careful to point out, this model of the air flow is an extreme idealization. One might expect that for results calculated on this basis to apply in practice, the actual velocity profile would have to admit a linear approximation to a distance of at least the order of a wavelength from the interface; and this requirement is certainly not met in experiments such as those of Knuth (1954), in which the air flow is turbulent and the slope of the profile varies very rapidly near the interface—whereas the waves observed are much longer than, say, the thickness of the viscous sublayer. Nevertheless, the linear-profile model has the unique merit that, while it is a physically *possible* example of viscous shearing flow, the respective form of the Orr–Sommerfeld equation (which determines the structure of a periodic disturbance) can be solved exactly. And so there is generally much to be said in favour of using this fairly manageable model as a first step towards clarifying physical problems concerned with shearing flows.

In § 5 we shall consider uniform shearing flow as a first example on which to try our general theory. The equations obtained in this example for the stresses on fixed or very slowly moving waves reduce to attractively simple approximate forms, whose compactness invites a generalization by Fourier's theorem. This is done in § 8, mainly to illustrate an interesting 'quasi-sheltering' effect (see three paragraphs below) depending on the phase relations between the stresses and the boundary displacement, an example being provided by a single-humped perturbation of the boundary shaped like the graph of $f(x) = (x^2+b^2)^{-1}$.

We next recall what has been done in the past to account for *turbulent* flow over waves. In attempts to explain water-wave formation several theories have been put forward which recognize in some way the turbulent character of the wind; but there are in general only two courses open towards workable theories, each of these having to start from a bold simplification of the physical problem. One of these is based on considerations quite different from the material of this paper: the effects of the waves on the air flow are neglected entirely, and attention is fixed on the response of the water surface to random fluctuations of normal pressure, which are taken to be the same as on a plane surface.†

The alternative course which has been followed by a number of authors interested in water-wave formation is to deal with the turbulent flow in much the same way as we shall. The turbulence is neglected except inasmuch as it may determine relevant properties of the mean shear flow, and attention is fixed on effects arising from the wavy disturbance of the air stream. The best-known example of this approach is the sheltering theory of Jeffreys (1924, 1925), which makes use of the principle that an already existing wave-train will grow if the wind supplies energy to it at a rate greater than that of viscous dissipation in the water. Inertial effects and tangential stresses due to the wind are neglected; and the only

† Phillips (1957) has published a new theory on these lines, giving an illuminating account of water-wave generation by a random pressure distribution.

property of the normal pressure distribution needed for the calculation of the average energy supply is the Fourier component in phase with the wave slope. This component is supposed to arise as the flow separates from the leeward side of each wave, causing the pressure there to be lower than on the windward side; and, to express its magnitude, a sheltering coefficient s is introduced. Jeffreys's theory provides no estimate of s, and was for a long time held in doubt since experimental estimates of s by use of fixed wave models (e.g. Motzfeld 1937) give values much too small to explain observed rates of wave growth.

However, this objection has been removed by the recent work of Miles (1957), who showed theoretically that even when there is no separation of the flow (i.e. no sheltering in the sense in which this term is usually understood) sufficiently large values of s can occur when there is a critical point away from the wave surface in a region of the velocity profile where the curvature is large negatively. The development of a substantial pressure component in phase with the wave slope is related to the phase discontinuity in the longitudinal velocity which is well known to occur across the friction layer surrounding the critical point (Tollmien 1929). Thus, the value of s for a moving wave may greatly differ from that for a similar fixed model, which has the critical point right at the boundary. It is remarkable that the 'quasi-sheltering' effect discovered by Miles is indicated by linearized perturbation theory, and is therefore quite distinct from the essentially non-linear effect (i.e. the formation of a wake) implied by the term sheltering. For the velocity profile, Miles took the 'universal' logarithmic law applicable to turbulent boundary layers at very large Reynolds numbers.

Miles's paper appears to have greatly invigorated the 'stability theory' of wave generation by turbulent wind† (the description 'sheltering theory' now seems inappropriate); and the present investigation owes largely to the stimulus of his work. In part, the following analysis will amplify Miles's treatment of disturbed turbulent profiles: for instance, all the surface stresses will be estimated, and the significant effect of viscosity at the wave surface will be examined. We shall see that both the normal and tangential stress distributions may in some circumstances be such as to do work on a moving wave, and their effectiveness in this respect will be compared: this property will be disclosed for a linear profile as well

† It seems evident that the two types of theory of which respectively the contributions of Phillips and Miles are the most advanced examples are both relevant to natural cases of wave generation by wind, despite the greatly different character of the two. They can be regarded as alternatives for *ad hoc* application according to the nature of the observed waves. For instance, if waves are seen to develop which are fairly regular and long-crested and which travel much slower than the wind, then clearly a stability theory is the more likely to be useful (and the present work a possible help). To mention just one instance where observation suggests that effects of the disturbed mean flow predominate over effects of the turbulence, the author has noticed that when a turbulent air-stream blows over a thin layer of highly viscous liquid like syrup the free surface may not be visibly agitated by the turbulent pressure fluctuations acting upon it; but when the wind speed is increased beyond a certain limit, regular long-crested waves of the order of 1 cm in length suddenly appear. When a film of water is observed in the same circumstances, some apparently random agitation is always discernible. Nevertheless, long-crested and fairly slow-moving waves arise at a wind speed rather less than before; and in all respects except the smoothness of the surface, their character seems much the same as of the waves on the more viscous liquid.

as for boundary-layer profiles, although the former case is special in that viscous effects at the boundary are primarily responsible for 'quasi-sheltering'. In each case, these viscous effects cause 'quasi-sheltering' when the boundary is rigid (i.e. when there is no critical point away from the boundary, so that the mechanism considered by Miles is absent).

To conclude this introduction, we must refer again to the matter of flow separation from the wave surface, which is probably the most vital consideration of all regarding the practical usefulness of the present linearized theory. It is to be expected that any actual flow at fairly high Reynolds number will separate if the waves are made steep enough or the wave-train long enough; but the value of our analysis rests on the supposition that the flow will remain attached to a fairly short wave-train of reasonably small steepness. In this respect applications to turbulent flows are more secure than to laminar flows, which tend rather readily to separate when given a wavy disturbance (Quick & Schröder (1944) showed this theoretically by use of numerical methods). The experiments of Motzfeld (1937) are a source of encouragement here. He in turn fixed four different rigid wave models in a wind tunnel, and measured pressure and velocity distributions throughout the whole flow—which was fully turbulent. The results obtained with the first model, which was a sinusoidal wave-train of 0·75 cm amplitude and 30 cm wavelength, are of particular interest at present. The pressure distribution over the wave surface was found to be very approximately sinusoidal, as it should be according to our theory; and no evidence of separation was found. The next two models were steeper waves, the ratio of wavelength to amplitude being for both about 20; and the pressure distribution was found to be slightly skew in the windward direction. However, there was still no evidence of separation, as may be seen from Motzfeld's diagrams of the streamlines following the contour of the waves. The fourth wave model was sharp-crested; and the flow was observed to separate from the wave crests and re-attach close to the troughs. A comparison between some of Motzfeld's measurements on his first wave model and results from the present theory will be made in § 7, a reasonable agreement being found.

It must be noted that Stanton, Marshall & Houghton (1932) also measured pressure distributions over wave models in a wind tunnel and found results markedly different in character from those of Motzfeld: the discrepancy has been commented upon by Ursell (1956). For their first series of measurements, the model was a train of twenty-seven sine waves, whose amplitude and wavelength increased linearly with distance but kept a constant ratio of about 0·1: the length of the first wave was 5·1 cm, and that of the last was 21·6 cm. The pressure distributions were measured over the tenth and twenty-seventh waves for various wind speeds. For their second series of measurements, two models were used on which the waves were of constant wavelength, the ratio of amplitude to wavelength being about 0·2. Pressure distributions were measured at 40 and 80 wavelengths from inlet, and were found to be much the same in the two places. These various observations differed from Motzfeld's in that the pressure distributions were very irregular, being not even roughly sinusoidal: for instance, Fourier analysis of a typical distribution shows the amplitude of the second harmonic to be of the same order of magnitude as the fundamental amplitude. Thus, it seems evident that in these experiments there was turbulent boundary-layer separation from the wave profiles. That this should have occurred here and not in Motzfeld's experiments with a sinusoidal profile is readily accountable to the fact that the waves were considerably steeper (particularly in the second series of measurements), and also to the fact that the wave-train was much longer (Motzfeld's results considered in § 7 were obtained from a model only three wavelengths long). These considerations suggest that the present theory should preferably be restricted to waves with an amplitude to length ratio less than, say, about 0·03 or 0·02 and with fairly short fetch.

2. Formulation of the problem

We shall express all variables in non-dimensional form, implying that the units of length and of velocity are to be taken as a certain length l and velocity U_0 characteristic of the physical problem: thus, for instance, any symbol representing a velocity is to be interpreted finally as a multiple of U_0. Time is made dimensionless on the understanding that l/U_0 is the unit; and stresses are considered as multiples of ρU_0^2, ρ being the density of the fluid in question. The Reynolds number $R = U_0 l/\nu$, where ν is the kinematic viscosity, becomes an important parameter in this problem since the effects of viscosity will not be ignored.

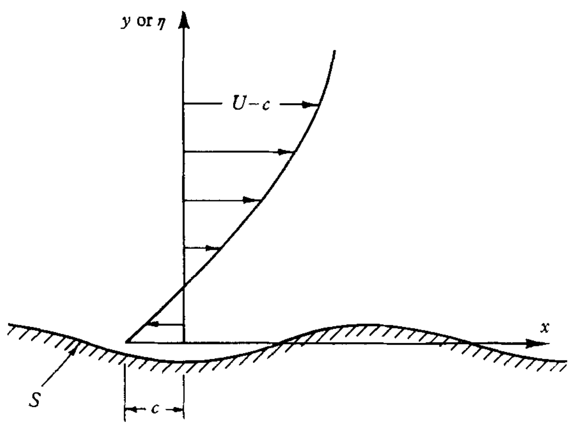

FIGURE 1. Definition sketch showing the undisturbed velocity profile as it would appear to an observer moving with the wave.

It is convenient to use a frame of reference in which the wave upon the bounding surface S is stationary. Thus, with respect to Cartesian axes (x, y) taken as in figure 1 and moving at speed c with the wave, the velocity parallel to x in the primary flow above the surface S is $U(y) - c$, where $U(y)$ is the velocity relative to the material surface (which in this frame of reference slips to the left with speed c). The equation of S is taken to be

$$y = a\, e^{ikx} \tag{2.1}$$

with the understanding that the real part represents the physical boundary. The amplitude a is assumed to be small compared with the wavelength $2\pi/k$, so that $(ka)^2$ is negligibly small; and this is intended to be our only restriction on the size of the wave. However, a difficulty now appears on consideration of the boundary conditions which apply at S. Large velocity gradients are to be expected right at this boundary (i.e. $U'(0)$ will be large), so that the trace of equation (2.1) upon the primary velocity profile may represent large velocity variations notwithstanding the present assumption about the smallness of a. But two boundary conditions relating respectively to the normal and tangential velocity components must be satisfied at S. It would seem therefore that a linearized theory along the lines so far suggested would require for accuracy a further restriction on the wave amplitude, namely, that a be small enough for the curve (2.1) to be confined to a region over which the variation of U is small. For instance, a would need to be considerably less than the width of the viscous sublayer in a turbulent boundary layer. This restriction would be too severe for a realistic theory; but fortunately it can be avoided by a slight change in approach to the problem.

A way out of the difficulty is suggested by a well-known property of boundary layers along curved walls (see, for example, Goldstein 1938, p. 119; or Schlichting 1955, p. 98). In a region where the wall curvature is fairly small and there is no large adverse pressure gradient, the changes in the boundary-layer profile at different positions along the wall are not large if measured relatively to the wall, being in fact of the first order in the curvature. In other words, the flow tends to follow the contour of the wall in such a way that the main features of the boundary layer, which may include a sharp velocity gradient right at the wall, are largely preserved—or at least undergo no more than first-order changes in a reasonable distance. Now, if for the present problem use is made of a curvilinear orthogonal system of co-ordinates in which the wave S is a co-ordinate line, the disturbed flow can be described by a perturbation representing only the *difference* between the actual flow pattern and the pattern formed by 'bending' the primary profile to follow the wave (i.e. the latter is simply $U(\eta)$, where η is the curvilinear co-ordinate perpendicular to S). This perturbation will be of the order of ka at most; and since the boundary conditions on it will not entail an 'overlapping' of the primary flow pattern as before, there need be no additional restriction on the magnitude of a in order to satisfy these boundary conditions with sufficient accuracy. The reasoning outlined here will perhaps be made a good deal clearer by what follows.

We take orthogonal co-ordinates (ξ, η) defined by

$$\left.\begin{aligned} \xi &= x - ia\, e^{-k(y-ix)}, \\ \eta &= y - a\, e^{-k(y-ix)}, \end{aligned}\right\} \tag{2.2}$$

in terms of which the equation of S is, to the first order in ka, simply $\eta = 0$. The Jacobian of this transformation is

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = 1 + 2ka\, e^{-k(\eta - i\xi)} \tag{2.3}$$

to the first order. Note that incidentally ξ and η are the same as the velocity potential and stream function for irrotational wave motion in an inviscid fluid (cf. Lamb 1932, § 233).

Let ψ denote the stream function for the present case, its form in the absence of waves (i.e. $a = 0$, $\eta = y$) being

$$\psi_0 = \int_0^\eta \{U(\eta) - c\}\, d\eta.$$

For the disturbed flow, ψ may be expressed in the form ψ_0 plus a periodic perturbation proportional to a: thus we write

$$\psi(\xi, \eta) = \psi_0(\eta) + a\{F(\eta) + [U(\eta) - c]\, e^{-k\eta}\}\, e^{ik\xi}, \tag{2.4}$$

where the function $F(\eta)$ has yet to be determined. (The inclusion in (2.4) of the term with $(U-c)$ as a factor simplifies the ensuing work to some extent; it can be included arbitrarily since $F(\eta)$ is unspecified.) Following from (2.4), the velocity components parallel to ξ and η are given respectively by

$$\left.\begin{aligned} u &= J^{\frac{1}{2}}\psi_\eta = U - c + a\{F' + U'\, e^{-k\eta}\}\, e^{ik\xi}, \\ v &= -J^{\frac{1}{2}}\psi_\xi = -ika\{F + (U-c)\, e^{-k\eta}\}\, e^{ik\xi}, \end{aligned}\right\} \tag{2.5}$$

where the subscripts denote partial differentiations, and the vorticity is

$$\zeta = J\{\psi_{\xi\xi} + \psi_{\eta\eta}\} = U' + a\{F'' - k^2F + U''\, e^{-k\eta}\}\, e^{ik\xi}. \tag{2.6}$$

Consider now the boundary conditions to be satisfied by the velocity components at S, that is, at $\eta = 0$. If S is a solid boundary (which need, of course, be flexible if c is not to be zero), a requirement for the primary flow is that $U(0) = 0$. Two boundary conditions then apply to the disturbed velocity components expressed by (2.5). First, the normal velocity v at S must be zero since the wave is stationary; and hence (2.5) shows that

$$F(0) = c. \tag{2.7}$$

Secondly, the tangential velocity u must satisfy a condition of non-slipping relative to solid particles fixed in S; and this requires that

$$F'(0) = -U'(0), \tag{2.8}$$

which follows from (2.5) on consideration that, at the wave surface, $u = -c$ to the first order of approximation in ka.

These boundary conditions can easily be modified to include the case where S is the boundary of another fluid, so that the tangential velocity along S may vary. Clearly, by adjusting the frame of reference the velocity of the undisturbed interface can be put equal to $-c$, i.e. $U(0) = 0$ as before: the velocities in the second fluid must then take negative values in order that the shearing stress due to the primary flow should be continuous across the interface. It may also be apposed that the variable component of the tangential velocity can always be expressed in the form $\beta a\, e^{ik\xi}$. Accordingly, the boundary condition (2.7) remains unchanged; but (2.8) is replaced by

$$F'(0) = -U'(0) - \beta. \tag{2.9}$$

The cases now covered may be summarized as follows:

(i) The boundary is solid and rigid; thus $c = 0$ and $\beta = 0$.

(ii) It is solid but flexible (e.g. an inextensible membrane); thus a wave can be propagated along it with finite c, but $\beta = 0$.

(iii) It is an interface between fluids; thus both c and β may be non-zero, and are likely to depend on the properties of the second fluid.†

The validity of these boundary conditions depends only on ka being small: that is, they are consistent with the linearized approximation to ψ expressed in (2.4). The difficulty mentioned earlier has therefore been avoided, which is due essentially to the fact that the expressions (2.5) for the velocities involve $U(\eta)$ rather than $U(y)$ as they would if developed in Cartesian co-ordinates: in the latter case the boundary conditions would be applied on $y = a\,e^{ikx}$ instead of $\eta = 0$. For a boundary-layer flow, $U'(0)$ will become infinite in the limit as the Reynolds number $R \to \infty$; but we can expect the present method of linearization to remain valid at indefinitely high Reynolds numbers, since $R \to \infty$ is precisely the condition for vorticity to be constant along streamlines, e.g. the streamline $\eta = 0$.

Expressions will now be found for the stresses acting upon S. Their derivation is slightly more complicated in curvilinear rather than Cartesian co-ordinates; but there is no need to go into details of the derivation here, since the calculations require merely a straightforward application of the theory of general orthogonal co-ordinates as is explained in many text-books (e.g. Goldstein 1938, § 39). It will be convenient to express the stresses in terms of ψ rather than of the velocity components.

Consider first the shearing stress τ, which according to a well-known result is the same as $R^{-1}(\psi_{yy} - \psi_{xx})$, where R is the Reynolds number. (This is, of course, the dimensionless form of τ implied by the remarks at the beginning of this section.) An alternative expression involving derivatives of ψ with respect to ξ and η is found to be

$$\tau = R^{-1}\{(\xi_x^2 - \xi_y^2)(\psi_{\eta\eta} - \psi_{\xi\xi}) + 4\xi_x\xi_y\psi_{\xi\eta} - 2\eta_{xy}\psi_\xi + 2\xi_{xy}\psi_\eta\}. \tag{2.10}$$

Substituting (2.2) and (2.4) into (2.10) and omitting second-order terms, we get

$$\tau = R^{-1}[U' + a\{F'' + k^2F + U''\,e^{-k\eta}\}\,e^{ik\xi}]. \tag{2.11}$$

This is evaluated at $\eta = 0$ to give the surface shearing stress, say τ_s. The first term in the expression for τ_s is $R^{-1}U'(0)$, which is the stress exerted by the

† Case (iii) needs special qualification when we apply the theory to turbulent flows. Turbulent fluctuations in the first fluid will always in some degree be transmitted to the second; and if the densities and viscosities of the two fluids were not much different, Reynolds stresses would have a predominant effect on the interface, and there would be no viscous sublayer as in the case of a solid boundary. But in the following analysis we need to assume a viscous layer at the boundary; and so in case (iii) we imply that the second fluid is comparatively very viscous and dense (e.g. water if the first fluid is air). It is then reasonable to assume that the flow has a structure similar to one over a solid boundary, including a definite sublayer. In fact, for the case of air–water, the main effect of turbulence in the water would appear to be only the resulting roughness of the surface, whose equivalent 'roughness length' affects the air profile in the well-known way (e.g. see Miles 1957). Therefore, provided the surface agitation is mild enough for the air flow to remain 'aerodynamically smooth' (i.e. the sublayer thickness exceeds the height of the random surface disturbances), the mobility of the boundary remains a factor of secondary importance in relation to the structure of the air flow.

undisturbed primary flow. If the variable part of τ_s is denoted by $T_s a\, e^{ik\xi}$ and the boundary condition (2.7) is introduced, then the factor T_s is given by

$$T_s = R^{-1}\{F''(0) + k^2 c + U''(0)\}. \tag{2.12}$$

The component of direct stress in the η-direction is expressible in the form

$$-p + 2R^{-1}J^{-1}(\xi_y^2 - \xi_x^2)\{(\xi_x^2 - \xi_y^2)\,\psi_{\xi\eta} + \xi_x\xi_y\psi_{\xi\xi} + \eta_x\eta_y\psi_{\eta\eta} + \xi_{xy}\psi_\xi + \eta_{xy}\psi_\eta\},$$

where p denotes the pressure. For convenience gravity forces are neglected, being independent of the hydrodynamic forces of present interest; but it is obvious that if the y-direction is upward, the (dimensional) pressure variation on S due to gravity is simply $-\rho g a\, e^{ik\xi}$. On substituting (2.2) and (2.4) and evaluating this expression at $\eta = 0$, we find that the normal stress σ_s acting *downward* on S is given by

$$\begin{aligned}\sigma_s &= p_s + 2ikaR^{-1}F'(0)\, e^{ik\xi} \\ &= p_s - 2ikaR^{-1}\{U'(0) + \beta\}\, e^{ik\xi},\end{aligned} \tag{2.13}$$

where p_s is the pressure at S, and where the second equality follows from the boundary condition (2.9). Here the term in R^{-1} is completely known; but this term will appear later to be negligible within the overall scheme of approximation to be adopted.

To find p_s, we consider the Navier–Stokes dynamical equations expressed in terms of the co-ordinates (ξ, η) and with ψ and p as dependent variables. The two equations of motion respective to the ξ and η-directions are found to be

$$J(\psi_\eta\psi_{\eta\xi} - \psi_\xi\psi_{\eta\eta}) + \tfrac{1}{2}J_\xi(\psi_\xi^2 + \psi_\eta^2) = -p_\xi + R^{-1}\zeta_\eta, \tag{2.14}$$

$$J(-\psi_\eta\psi_{\xi\xi} + \psi_\xi\psi_{\xi\eta}) + \tfrac{1}{2}J_\eta(\psi_\xi^2 + \psi_\eta^2) = -p_\eta - R^{-1}\zeta_\xi, \tag{2.15}$$

in which ζ is to be related to ψ by equation (2.6). The substitution of (2.3), (2.4) and (2.6) into (2.14) and (2.15) leads to two alternative equations for p in terms of F and its derivatives. The pressure variations due to the wave disturbance may be considered distinctly from those in the primary flow, so that we may put $p = P(\eta)\, a\, e^{ik\xi}$ and therefore $p_\xi = ikp$. Hence, the first equation for p (obtained from (2.14)) gives

$$P = U'F - (U-c)\,F' - i(kR)^{-1}\{F''' - k^2F' + (U''' - kU'')\, e^{-k\eta}\}. \tag{2.16}$$

Alternatively, P can be found by an integration of (2.15) with respect to η, using the fact that the disturbance of the flow vanishes for $\eta \to \infty$. The result then is

$$P = k^2\int_\eta^\infty (U-c)\,F\,d\eta - i(kR)^{-1}\left\{k^2F' + \int_\eta^\infty (k^4F - k^2U''\, e^{-k\eta})\, d\eta\right\}. \tag{2.17}$$

Either of these results, which obviously must be equivalent, may be evaluated at $\eta = 0$ to give $P(0) = P_s$, say, the amplitude of the surface pressure p_s. However, this step is postponed until § 3 where the alternative expressions for P_s will be shown to have an interesting physical interpretation.

Since T_s and P_s may have real and imaginary parts, they express both the magnitudes of the respective stresses and their phases in relation to the wave on the boundary. As $F(0)$ and $F'(0)$ are determined by the boundary conditions, it is seen from (2.12) and (2.16) that the problem is reduced essentially to finding $F''(0)$ and $F'''(0)$. This is by far the most difficult part of the task.

At this point it is suitable to list some assumptions which can be made on physical grounds regarding the magnitudes of certain quantities arising parametrically in these calculations. The assumptions form the basis for various approximations which will be introduced later on; and there is some advantage in setting them out beforehand for reference. They can be expressed symbolically as follows:

$$\left.\begin{array}{ll}\text{(i)}\ kR \gg 1; & \text{(ii)}\ [kRU'(0)]^{\frac{1}{3}} \gg k; \\ \text{(iii)}\ U'(0) \gg \beta; & \text{(iv)}\ (kRc^3)^{\frac{1}{2}} \gg U'(0).\end{array}\right\} \tag{2.18}$$

The first three of these rest essentially on the single assumption of large Reynolds number; but (iv) is rather special, and will be assumed only when the critical point is considered to lie well away from the boundary. Condition (iii) is totally satisfied if S is a rigid or flexible solid boundary, and is quite amply satisfied if S is an interface with a comparatively viscous second fluid (e.g. water if the main fluid is air). The basis for these various assumptions will be fully examined in later parts of the discussion.

3. The function F

The elimination of P between (2.16) and (2.17) leads to the fourth-order equation for $F(\eta)$

$$(U-c)(F''-k^2F)-U''F = (ikR)^{-1}\{F^{\text{iv}}-2k^2F''+k^4F+(U^{\text{iv}}-2kU''')\,e^{-k\eta}\}, \tag{3.1}$$

which is seen to be reducible to the Orr–Sommerfeld equation by the omission of the terms in U on the right-hand side. It will now be shown that this omission is in fact justified. We note first that the assumption of a parallel primary flow implies, strictly speaking, that $U^{\text{iv}} = U''' = 0$ everywhere, since U must itself be a solution of the Navier–Stokes equations. However, we do not intend to confine the argument to velocity profiles of the strictly parallel sort (i.e. linear or parabolic ones), and we therefore assume that $U \equiv U(y)$ is an adequate approximation for many distributions, particularly of boundary-layer type, whose variation with x is small over several wavelengths. This assumption has commonly been introduced in theories of boundary-layer stability, and its validity amply confirmed (see, for instance, Lin 1955, § 5.1).

A more cogent reason for simplifying (3.1) is forthcoming when another familiar point from stability theory is recalled. Suppose that the primary distribution is characterized by a boundary layer whose thickness is selected as the reference length on which the Reynolds number R is based (cf. Schlichting 1955, p. 316). We note that U''' and U^{iv} are then not of a greater order of magnitude than U or U'', and also that by the boundary conditions F is of the same order as U or c. Hence, as the parameter $(kR)^{-1}$ multiplying the right-hand side of (3.1) is taken to be very small (assumption (i) in (2.18)), it is seen that this side of the equation plays an insignificant part in determining F, except in regions where F^{iv} becomes exceptionally large—certainly much larger than U^{iv} and U'''. One such region may occur very close to the boundary, where viscosity becomes important by virtue of the non-slipping condition; and the only other possibility is a region around a so-called 'critical point' at which $U = c$ and the 'inviscid' equation (i.e. (3.1) with the right-hand side zero) therefore has a singular point. Accordingly, the equation for F can be taken to be the Orr–Sommerfeld equation

$$(U-c)(F''-k^2F)-U''F = (ikR)^{-1}\{F^{\mathrm{iv}}-2k^2F''+k^4F\} \tag{3.2}$$

with the understanding that the right-hand side is strictly accurate only for linear or parabolic velocity profiles, but is negligible anyway except at 'friction layers'—where the most important viscous term $(ikR)^{-1}F^{\mathrm{iv}}$ is alone an adequate approximation to the right-hand side of (3.1).

Equation (3.2) has been studied extensively as part of stability theory, some details of which are immediately useful for the present problem. Note, however, that the present F is rather different from the dependent variable in the Orr–Sommerfeld equation as it occurs in stability theory: there y is usually the independent variable, here it is η.

We first recall some well-known properties associated with critical points. Let $\phi(\eta)$ be the solution of

$$(U-c)(\phi''-k^2\phi)-U''\phi = 0 \tag{3.3}$$

(i.e. the inviscid form of (3.2)). At a position $\eta = \eta_c$, where $U = c$, this equation has a singular point, and consequently ϕ ceases to be an approximate solution of the complete Orr–Sommerfeld equation, even with $(kR)^{-1}$ very small. Since $U''/(U-c)$ has a simple pole at $\eta = \eta_c$, the formal expansion of ϕ about $\eta = \eta_c$ involves a term in $(\eta-\eta_c)\log(\eta-\eta_c)$; and therefore the correct form of ϕ as an approximate solution away from the critical point is in doubt until the appropriate branch of the logarithm is decided. This ambiguity can only be resolved by consideration of the complete equation (3.2); in other words, account must necessarily be taken of the effects of viscosity in the vicinity of the critical point. Tollmien (1929) first demonstrated that if the logarithm is expressed as $\log(\eta-\eta_c)$ for $\eta > \eta_c$, it is to be replaced by $\log(\eta_c-\eta)-i\pi$ for $\eta < \eta_c$. (This matter is one of the most vital and frequently discussed topics in stability theory; and it can be said that the lucid account given originally by Tollmien is one of the major contributions to modern fluid mechanics.) When this adjustment is made, ϕ can confidently be taken as an approximate solution of (3.2) applying either side of the critical region, the error being known to be $O(1/kR)$. We shall adopt this approximation for cases where c is substantially greater than zero: that is, the critical point occurs well within the fluid. Attention is confined to velocity profiles in which U increases monotonically with η (i.e. profiles of boundary-layer type), so that there cannot be more than one critical point. For the case $c \doteqdot 0$ where the critical point occurs at or very close to the boundary, special treatment will be needed.

The inviscid solution ϕ may not be a valid approximation close to the boundary S, since viscosity has a significant effect there as a result of the boundary condition on the tangential velocity. Mathematically speaking, this also follows clearly from the fact that the solution of a second-order equation (2.3) cannot in general satisfy the condition that $\phi \to 0$ far from S and also two boundary conditions at S. Accordingly, we express an approximation to the complete solution F in the form

$$F(\eta) = \phi(\eta)+f(\eta), \tag{3.4}$$

where $f(\eta)$ is the appropriate rapidly varying solution of the full Orr–Sommerfeld equation: this must be the solution which diminishes rapidly with increasing η, since the other rapidly varying solution (which may be shown to increase in exponential fashion with η) is inadmissible physically (see Tollmien 1929; or Lin 1955, § 3.4 or § 8.5; or Schlichting 1955, p. 327). The region where $f(\eta)$ is significant may suitably be called the 'wall friction layer'.

The boundary conditions (2.7) and (2.9) can now be rewritten as

$$\phi(0)+f(0) = c, \tag{3.5}$$

$$\phi'(0)+f'(0) = -U'(0)-\beta; \tag{3.6}$$

and these expressions can be combined to give

$$\phi'(0)-[f'(0)/f(0)]\,\phi(0) = -U'(0)-\beta-c[f'(0)/f(0)], \tag{3.7}$$

which may be regarded as the boundary condition on ϕ. This will be examined in § 4. Note that if $f'(0)/f(0) \gg U'(0)/\phi(0)$, this condition is approximately $\phi(0) = c$, which is the form of the kinematical boundary condition which would arise if viscosity were neglected entirely. Note also that β may be neglected in (3.7) according to assumption (iii) of (2.18).

Let us now reconsider the expressions (2.16) and (2.17) for the pressure amplitude P. When evaluated at $\eta = 0$, (2.16) gives

$$P_s = k^2 \int_0^\infty (U-c)\,F\,d\eta + O(kR)^{-1}. \tag{3.8}$$

The terms which are $O(kR)^{-1}$ may be neglected according to assumption (i) of (2.18); and, as a further approximation (which in fact can be shown to be well within the limits of the previous one), F will be replaced by ϕ in the integral. The latter step can be taken because the integral is insensitive to the contribution from the wall friction layer—which is the only place where $F = \phi$ is not a good approximation (note that it is good through the friction layer about the critical point, even though the derivatives of ϕ cease to be valid approximations there). Thus we get

$$P_s = k^2 \int_0^\infty (U-c)\,\phi\,d\eta. \tag{3.9}$$

A result of this form might be obtained by neglecting viscosity from the start; but now viscosity can affect P_s through the boundary condition (3.7) on ϕ. The form of (3.9) suggests the attractive physical interpretation that the variable pressure at the boundary is, so to speak, generated by the cumulative action of the disturbance over the whole flow field, and is not particularly sensitive to the state of affairs near S. This is a familiar idea in boundary-layer theory. On the other hand, the form of (2.11) and (2.12) indicates that the surface shearing stress depends chiefly on circumstances close to the boundary (note F'' becomes much larger in the wall layer than elsewhere, due to the component f'').

An alternative expression for P_s follows from (2.17). When evaluated at $\eta = 0$, the first two terms on the right-hand side reduce to $-\beta c$ because of the boundary conditions; and we get

$$P_s = -\beta c + (ikR)^{-1}[F'''(0) + U'''(0) - kU''(0) + k^2\{U'(0) + \beta\}], \tag{3.10}$$

$$= (ikR)^{-1} F'''(0), \tag{3.11}$$

where (3.11) follows from the exact expression (3.10) because of assumptions (i) and (iv). A comparison of (3.11) with (3.9) shows $F'''(0)$ to be a large quantity, at least $O(kR)$; and clearly it is large due to the f component of F arising in the wall layer (in fact $F'''(0)$ can be replaced by $f'''(0)$ within the order of approximation implied in proceeding from (3.10) to (3.11)).

Let us consider equation (2.17) a stage further. We know from (2.16), and also as a familiar result from boundary-layer theory, that the pressure varies quite gradually with η; in particular, it does not change significantly over the thin wall layer. A good approximation to P_s will therefore be given by evaluating (2.17) just outside the wall layer, say at $\eta = \epsilon$: this takes the form

$$P_s = U'(\epsilon)\,\phi(\epsilon) - [U(\epsilon) - c]\,\phi'(\epsilon), \tag{3.12}$$

the terms in $(kR)^{-1}$ on the right-hand side of (2.17) being negligible outside the wall layer. But we have seen when deriving (3.11) that the terms independent of R in (2.16) (i.e. the terms like (3.12)) reduce at $\eta = 0$ to the small quantity $-\beta c$, and that a term negligible at $\eta = \epsilon$ becomes dominant at $\eta = 0$. In other words, the two groups of terms rapidly exchange role in keeping P approximately constant across the wall layer. This behaviour is perhaps obvious physically; but it is interesting to see mathematically how, when proper account is taken of viscosity, the pressure variation—an effect generated essentially by inviscid behaviour in the main flow—is transmitted to the boundary through the 'region of influence' of the non-slipping boundary condition, which would have to be ignored on the basis of at best some tentative physical argument if viscosity were neglected entirely. (Note, by the way, that (3.12) is exactly equivalent to (3.9) with the lower limit of integration replaced by ϵ: this relationship is proved by an integration of (3.3).)

Of the three approximate expressions which have been given for P_s, the first (3.9) would seem likely to be most generally useful, being of a form comparatively insensitive to errors in ϕ. Its intuitive appeal has already been remarked; and it indicates clearly the important fact, to be discussed fully later, that conditions at the critical point largely determine the *phase* of P_s. This was first demonstrated by Miles (1957), whose work will be recalled in some detail just below. As was mentioned in the discussion following equation (3.3), the inviscid solution undergoes a phase change across the critical point. The magnitude and sign of this change depend on the values of U' and U'' at $\eta = \eta_c$ (Tollmien 1929); and hence these values obviously must affect the phase of P_s as calculated from (3.9).

The imaginary part of P_s has an important physical significance, as it can be shown that in the absence of viscosity the rate of energy transfer from the fluid to unit area of the wave surface is $ka^2c\mathscr{I}\{P_s\}$. An estimation of this quantity is the vital step in 'sheltering' theories of wave generation by wind, in which the growth of water waves is supposed to occur when the energy supply exceeds the rate of dissipation by viscosity in the water (see Ursell (1956) for a review of this subject). Miles (1957) has published an interesting calculation of $\mathscr{I}\{P_s\}$ for velocity profiles characteristic of turbulent boundary layers, the calculation being based essentially on the evaluation of an expression similar to (3.12). His theory was developed in Cartesian co-ordinates; but the restriction implied by the presence of large velocity gradients—as explained in § 2—was recognized, and a way of avoiding it was suggested (p. 191 of his paper). The most important difference between the present model and that considered by Miles is the neglect of viscosity in his, so that the boundary condition on the inviscid solution ϕ needs to be a purely kinematic one applied at some distance from the wave surface—in contrast to the exact condition (3.7) which clearly depends on viscosity. In our notation the boundary condition assumed by Miles is equivalent to $\phi = -U+c$ at a surface approximately parallel to S but a little way from it, say $\eta = \epsilon$. This implies that $\eta = \epsilon$ is a streamline, which is not an obviously justifiable assumption *a priori*, particularly in view of the large velocity gradient near S. However, there is little cause to doubt the validity of Miles's theory in the particular examples treated by him. It will appear in § 4 that the inviscid boundary condition can be established rigorously as an approximation to the exact condition in some cases.

(It may be noted, incidentally, that if the behaviour of $\phi(\eta)$ implied by the assumption of an inviscid boundary condition (i.e. $\phi = c-U$ for η small) were to persist right to $\eta = 0$, the full viscous condition (3.7) would be satisfied—at least when β is neglected in comparison with $U'(0)$. At first sight this fact might seem to justify the inviscid condition as an approximation to the full viscous condition; but in fact this does not follow. Since $\phi = c-U$ is not an exact solution of (3.3), there is no assurance that this approximation satisfies (3.7) adequately because of the large parameter $f'(0)/f(0)$ in (3.7) which multiplies the error in ϕ.)

An important result due to Miles (1957) deserves to be recalled here. The opportunity will be taken to outline a derivation rather different from his. If the inviscid boundary condition noted above is accepted, equation (3.12) leads immediately to

$$\mathscr{I}\{P_s\} = [c-U(\epsilon)]\,\mathscr{I}\{\phi'(\epsilon)\}. \tag{3.13}$$

Now a property deducible from (3.3) (see Lin 1955, § 8.2) is that, for c real as at present, the quantity

$$W = \tfrac{1}{2}i(\phi\phi^{*\prime}-\phi^*\phi'),$$

where ϕ^* denotes the complex conjugate of ϕ, is constant everywhere except at the critical point ($U = c$) where it has a discontinuity. The exact form of the discontinuity can only be decided by a study of the complete Orr–Sommerfeld equation, which shows it to be

$$[W] = W(\eta_c^+)-W(\eta_c^-) = \pi(U_c''/U_c')\,|\phi_c|^2, \tag{3.14}$$

where the subscript c refers to values at the critical point. But $W = 0$ for $\eta > \eta_c$ since $\phi \to 0$ for large η; therefore $W(\epsilon) = -[W]$ if $\eta_c > \epsilon$. Also, owing to the boundary condition $\phi(\epsilon) = c-U(\epsilon)$ (i.e. $\phi = \phi^*$ at $\eta = \epsilon$), we have

$$W(\epsilon) = \tfrac{1}{2}i[c-U(\epsilon)]\,[\phi^{*\prime}(\epsilon)-\phi'(\epsilon)] = [c-U(\epsilon)]\,\mathscr{I}\{\phi'(\epsilon)\}, \tag{3.15}$$

which is the same as the right-hand side of (3.13). Hence

$$\mathscr{I}\{P_s\} = -\pi(U_c''/U_c')\,|\phi_c|^2. \tag{3.16}$$

This is the result found in a somewhat different way by Miles. It shows the energy supply from the fluid to be proportional to the curvature of the velocity profile at the elevation where $U = c$, the supply being positive (i.e. such as to promote the growth of waves) when the curvature is negative. However, it is clear that $\mathscr{I}\{P_s\}$ will not be given as simply as this when the exact boundary condition (3.7) has to be applied.

4. The wall friction-layer

We now consider the function $f(\eta)$, which can be regarded as an adjustment to the inviscid solution required near the boundary in order to satisfy the viscous boundary conditions. This idea is familiar from stability theory; but perhaps the clearest account of what it implies physically is that given by Lighthill (1953), who introduced it in another context. The wall friction-layer (i.e. the region where $f(\eta)$ is appreciable) should not, of course, be confused with the boundary layer by which the velocity profile U may be characterized: the wall layer is likely to be only a very small fraction of the whole boundary layer (see Lighthill 1953, p. 480).

If it is assumed that the effective thickness of the wall layer is very small compared with the wavelength $2\pi/k$, the equation (3.2) satisfied by $f(\eta)$ may be simplified by neglecting the terms in k^2. If it is further assumed that f varies much more rapidly than U, then the term $U''f$ can be neglected in comparison with $(U-c)f''$. Also, $\eta U'(0)$ can be taken as an approximation to $U(\eta)$ over the wall layer. (The fact that $U''(0) = 0$ if the primary flow has no pressure gradient provides further justification for these approximations; but clearly this is not essential.) Thus, the equation for f becomes

$$f^{\text{iv}} = ikR\{\eta U'(0) - c\}\,f''. \tag{4.1}$$

The various approximations which have been introduced can best be justified *a posteriori* by considering the properties of the solution of (4.1) (see § 8). The assumption which probably has greatest need of being tested in particular applications is the linear approximation for U over the wall layer. It will be shown in § 8 that this assumption is well justified in the case of laminar boundary-layer profiles. It is less secure, however, for turbulent boundary layers; because, although the wall layer is much thinner than in corresponding laminar cases, U varies very much more rapidly near the boundary. In effect, applications to turbulent boundary-layer profiles require that the wall layer should lie within the viscous sublayer, over which U is well known to vary linearly.

The 'inviscid' solution $a + b\eta$ of (4.1) is irrelevant here, being in fact equivalent to a linear approximation for $\phi(\eta)$ near $\eta = 0$. The solution of (4.1) which approaches zero rapidly for large η is expressible in the form

$$f(\eta) = A\int_C \exp\left[\tfrac{1}{3}t^3 + im\{\eta - c/U'(0)\}t\right]t^{-2}\,dt, \tag{4.2}$$

in which $m = [kRU'(0)]^{\frac{1}{3}}$, A is an arbitrary constant, and the path of integration in the complex t-plane is terminated at $-\infty$ and $\infty e^{\frac{1}{3}\pi i}$. The form of C in finite parts of the plane is immaterial except that C must pass above the origin; but it is most convenient to trace C along the two radii C_1 and C_2 indicated in figure 2, the singularity at $t = 0$ being skirted by a small arc as shown in the figure. In discussing this integral, the notation $z = m\{\eta - c/U'(0)\}$ will be useful.

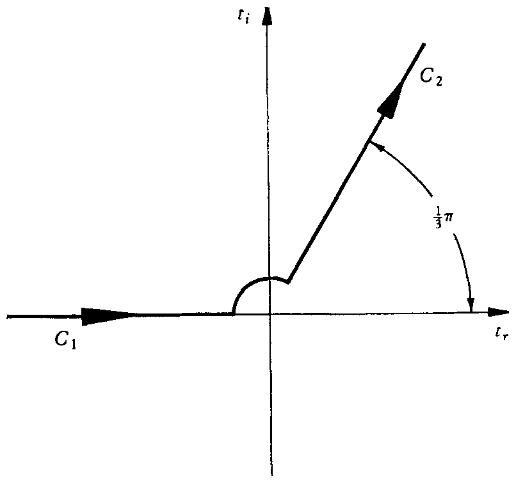

FIGURE 2. The path of integration in the complex t-plane.

Numerical values of this function have been calculated by several authors interested in stability theory, the usual course being to consider $-z$ as the independent variable. (This is the function usually denoted by ϕ_3 in papers on stability.) The values given by Holstein (1950) appear to be the most accurate, and a useful graph of the function appears as figure 4 in his paper (the abscissa there is equivalent to our $-z$). This figure shows both the real and imaginary parts of f to oscillate for increasing values of the argument z, but to decrease at an enormous rate over all ranges of z; for instance, their magnitudes decrease by factors which are $O(10^{-3})$ as z goes from 0 to about 4. Thus, we may consider the length m^{-1} as a measure of the effective thickness of the wall layer; and the assumption that this thickness is small compared with wavelength can therefore be expressed $m \gg k$ (i.e. assumption (ii) of (2.18)). It may be noted also that this solution is expressible in terms of Bessel functions (e.g. see Lin 1955, § 3.6; or Schlichting 1955, p. 327).

Let us consider how the quantity z will vary over the wall layer in some particular physical examples. First, for a rigid solid boundary we have $c = 0$; and so $z = m\eta$ is zero at the boundary and becomes positive away from it. The value of $f(\eta)$ will become negligibly small compared with its value on the boundary as $m\eta$ increases beyond, say, 1 or 2. Values of $f(0)$, $f'(0)$ and $f''(0)$ as required in the present analysis can be obtained very easily from (4.2) in this case.

Now suppose that c has a positive value small enough for the critical point $\eta = \eta_c$ to lie within the region next to the boundary over which the linear approximation $U = \eta U'(0)$ applies. Then $\eta_c = c/U'(0)$, and so $z = -m\eta_c$ at the boundary.

In fact, z increases from this negative value to zero at the critical point, and becomes positive for $\eta > \eta_c$. The viscous solution $f(\eta)$ diminishes rapidly as before, and will reach a quite insignificant magnitude at $z = 0$ if the ratio $mc/U'(0)$ $(= m\eta_c)$ is much larger than unity: i.e. the critical point lies well outside the wall layer. On the other hand, the critical point will occur in the wall layer if this ratio is of the order of unity. Now, even if the critical point lies outside the region where $U = \eta U'(0)$, the approximation for $f(\eta)$ is still valid provided that $mc/U'(0)$ is large; for, though $\eta = c/U'(0)$ is no longer the critical point, the wall layer is covered effectively while z still has large negative values, i.e. $f(\eta)$ has practically vanished well before z increases to zero. We remark that in this respect our use of the function $f(\eta)$ differs from the usual course in stability theory: there it is generally assumed that a linear approximation holds between the critical point and the (plane) wall, and in fact $U'(\eta_c)$ rather than $U'(0)$ is commonly written in the definitions of parameters corresponding to our m (cf. Lin 1955, p. 40).

A possible limitation of the present analysis must now be mentioned. The difficulty might be expected to arise in applications to turbulent boundary-layer profiles which have exceedingly high values of $U'(0)$. It appears feasible that the parameter $mc/U'(0) = (kR)^{\frac{1}{3}}c[U'(0)]^{-\frac{2}{3}}$ could be quite small even when the critical point occurred well away from the boundary. According to the line of argument considered above, the approximation (4.2) would not apply in this case. To meet an objection of this sort, it is necessary to examine the relation between $U'(0)$ and R for actual boundary layers. This step is deferred until § 8; and we meanwhile introduce the special assumption—to be justified there—that the parameter in question is reasonably large for all applications where the critical point is outside the region where $U = U'(0)\eta$, e.g. outside the viscous sublayer of a turbulent boundary layer. This assumption is listed as item (iv) in (2.18).

The quantity $f'(0)/f(0)$ appearing in (3.7) has special interest at present since it determines how, through the boundary conditions, viscosity affects the 'inviscid' solution $\phi(\eta)$ representing the disturbance in the main body of the fluid. In the literature on stability theory, this quantity is commonly expressed in terms of a certain function often named after Tietjens, who first introduced it: thus, in our notation

$$\frac{f'(0)}{f(0)} = -\frac{m}{D(-mc/U'(0))}, \tag{4.3}$$

where $D(\ )$ is the Tietjens function according to the usual definition. (Apparently many writers on stability have kept the notation D originally used by Tietjens, although F is used for $D \div z_0$ by Lin (1945, Part I; 1955, § 3.6).) At least six authors have given numerical tables of the real part D_r and imaginary part D_i of this function. The values given by Holstein (1950) cover the widest range of the argument; and a useful feature of his table is that the values calculated by five previous authors are included.

Graphs of $D_r(-z_0)$ and $D_i(-z_0)$ are shown in figure 3. It is seen that for z_0 greater than about 8 both functions cease to oscillate and rapidly approach the asymptote $1/\sqrt{(2z_0)}$ (see equation (4.13) below). The range of positive z_0 for which D_i is negative has special interest in a later part of our discussion. This is

$0 < z_0 < 2{\cdot}3$, the upper limit having been estimated from values of D_i tabulated by Holstein (1950) and Lin (1945). (Holstein includes values for $z_0 = 1{\cdot}5, 2{\cdot}0, 2{\cdot}5$; and Lin gives values (actually of his $F \equiv D/z_0$) at intervals of 0·2 for z_0. The latter are consistent with Holstein's, except that Lin's values for $z_0 = 2{\cdot}4$ and $2{\cdot}6$ seem to be incorrect.)

Figure 3 indicates that, for a given $m, f'(0)/f(0)$ becomes progressively larger as c gets fairly large. And so, recalling the remarks made below (3.7), we see that the exact boundary condition on ϕ tends to assume the simple form $\phi(0) = c$ which might be proposed as the (kinematical) boundary condition if viscosity were

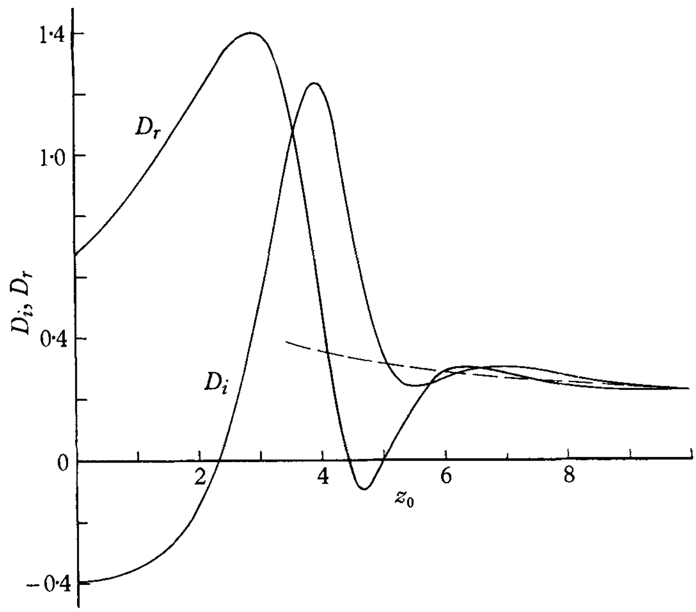

FIGURE 3. Graphs of the real and imaginary parts of the Tietjens function $D(-z_0)$. The dashed line is part of the curve $(2z_0)^{-\frac{1}{2}}$, which is the asymptote of D_r and D_i.

neglected from the start. The fact that for large wave velocities the effects of friction at the wave surface cease to be significant is, of course, to be expected on physical grounds (cf. Miles 1957, bottom of p. 187). However, it should be noted that as a criterion for neglecting viscosity a Reynolds number based on wave velocity and wavelength is not really adequate: the parameter $mc/U'(0)$ seems to provide a more reliable test, particularly in view of the fact that $U'(0)$ increases with Reynolds number in any self-consistent model for the flow.

We shall next consider the specific form taken by the boundary condition (3.7) in some special cases. To evaluate $f'(0)/f(0)$ explicitly, we might justifiably resort to existing tabulations of the Tietjens function; but it seems worth while to proceed directly from (4.2) for a few illustrations.

The case $c = 0$

Here the critical point coincides with the boundary. It will be assumed that there is no pressure gradient in the primary flow, so that $U''(0) = 0$: hence, as (3.3) now does not have a singular point at $U = c$, ϕ and ϕ' remain valid approximations over the whole flow. We also take $\beta = 0$ as the boundary is implied to be solid in this case. To find $f(0)$ explicitly from (4.2), the component integrals along the two radii shown in figure 2 are reduced to the same real integral by the substitutions $t = -s$ along C_1 and $t = e^{\frac{1}{3}\pi i} s$ along C_2. Thus we get

$$f(0) = A(e^{-\frac{1}{3}\pi i}+1)\left\{\int_{s^+}^{\infty} e^{-\frac{1}{3}s^3}\frac{ds}{s^2} - \frac{1}{s^+}\right\},$$

in which $s^+ \to 0$. An integration by parts now leads easily to

$$f(0) = -A3^{\frac{1}{6}}\Gamma(\tfrac{2}{3})\, e^{-\frac{1}{6}\pi i}. \tag{4.4}$$

The integral expression for $f'(0)$ obtained from (4.2) has a rather different character, as the contributions from C_1 and C_2 cancel, leaving as sole contribution the residue from the $120°$ circuit of the simple pole of the integrand at $t = 0$. It is found without difficulty that

$$f'(0) = A\tfrac{2}{3}\pi m, \tag{4.5}$$

and hence

$$f'(0)/f(0) = -1{\cdot}288\, e^{\frac{1}{6}\pi i}\, m. \tag{4.6}$$

The fractional error in this result (due to our neglecting terms in k^2 when deriving (4.1)) can be shown to be $O(k/m)^2$, which is very small according to assumption (ii) of (2.18).

Incidentally, a result which will be needed later can be found in the same way from (4.2): this is

$$f''(0)/f(0) = 1{\cdot}372\, e^{\frac{1}{3}\pi i}\, m^2. \tag{4.7}$$

The substitution of (4.6) into (3.7) now shows that the boundary condition to be satisfied by the inviscid condition is

$$\phi'(0) + 1{\cdot}288\, e^{\frac{1}{6}\pi i}\, m\phi(0) = -U'(0). \tag{4.8}$$

This covers both the kinematical and viscous conditions at the wave surface, and appears at first sight quite unlike any boundary condition that might be applied if viscosity were to be neglected from the start. The implications of this boundary condition will be examined later when $\phi(\eta)$ has been established explicitly; but this is a suitable place to note an interesting interpretation given by Lighthill (1953), who investigated the wall friction-layer in the course of his work on the upstream influence of boundary layers in supersonic flow.

To bring (4.8) into line with Lighthill's argument, the equation is multiplied by $(1{\cdot}288\, e^{\frac{1}{6}\pi i}\, m)^{-1}$, which may be denoted by $0{\cdot}776L$, say, where L has the same meaning as in Lighthill's notation. Thus, (4.8) becomes

$$\phi(0) + 0{\cdot}776L\phi'(0) = -0{\cdot}776LU'(0). \tag{4.9}$$

Now L is a complex parameter whose magnitude is a length comparable with the thickness of the wall layer; and over such a length from the wall the linear approximation $\phi(\eta) = \phi(0) + \eta\phi'(0)$ is certainly adequate. Hence, (4.9) has the interpretation that $\phi(\eta) = -U(\eta)$ approximately for $\eta = 0{\cdot}776L$, which could be taken to mean that the flow satisfies a kinematical condition on a certain surface $\eta = 0{\cdot}776L$ a small distance from the boundary $\eta = 0$. In other words, the effect of the wall layer appears to be as if the surface $\eta = 0{\cdot}776L$ were replaced by a solid wall, outside which viscosity had no effect.

It must be emphasized, however, that the equation $\eta = 0{\cdot}776L$ does *not* represent any surface in the physical plane. For, like the other complex parameters in this analysis, L is merely an operator carrying phase as well as magnitude significance, and obviously has no physical meaning divorced from the connotation of operating on periodic functions of ξ. This hypothetical wall has therefore to remain an abstract concept.

The case $mc/U'(0) = O(1)$ or less

The critical point now occurs away from the boundary yet still within the friction layer. This case is exemplified by waves upon a film of highly viscous liquid dragged along a plane wall by an air stream: the wave speed would be of the order of the mean velocity of the film, which would be only a very small fraction of the velocity in the main air stream. However, this case may still apply when c is a significant fraction of the free-stream velocity, if the initial velocity gradient is large as in turbulent flows with a boundary layer.

It would be appropriate here to use the expression (4.3) in terms of the Tietjens function. But, to bring out the effects of small yet finite values of $mc/U'(0)$, a useful alternative might be to approximate the integrals for $f(0), \ldots, f^n(0)$ by the leading terms of Taylor series in powers of this parameter. Accordingly, (4.2) gives

$$f^n(0) = A[I_n - \{imc/U'(0)\}\, I_{n+1} + \tfrac{1}{2}\{imc/U'(0)\}^2 I_{n+2} + \ldots], \qquad (4.10)$$

where $$I_N = \int_C e^{\frac{1}{3}t^3}\, t^{N-2}\, dt \quad \text{with} \quad N = 0, 1, 2, \ldots.$$

This scheme of approximation leads to

$$f'(0)/f(0) = -1{\cdot}288\, e^{\frac{1}{6}\pi i} m[1 - 0{\cdot}223\, e^{\frac{1}{6}\pi i}\{mc/U'(0)\} + \ldots]. \qquad (4.11)$$

Obviously, the results for $c = 0$ apply approximately to this also if $mc/U'(0)$ is small enough.

The case $mc/U'(0) \gg 1$

The critical point now occurs well outside the wall layer. We first remark that, as $f'(0)/f(0)$ gets very large in this case, the term $U'(0)$ in the boundary condition (3.7) takes on secondary importance. Hence, according to assumption (iii) of (2.18), the term β may be neglected entirely. This boundary condition may therefore be rewritten

$$\phi(0) - c = \{f(0)/f'(0)\}\{\phi'(0) + U'(0)\}. \qquad (4.12)$$

Near the boundary the exponent z in (4.2) is large negatively; and so it is suitable to use an asymptotic approximation to the integral—and to similar integrals for f' and f''. These may be obtained by the method of steepest descents. In this way we obtain (cf. Lin 1945, p. 136; 1955, § 8.5)

$$f'(0)/f(0) \sim -e^{-\frac{1}{4}\pi i}(kRc)^{\frac{1}{2}}, \qquad (4.13)$$

and $$f''(0)/f(0) \sim -ikRc. \qquad (4.14)$$

These two formulae are reliable for $mc/U'(0)$ greater than about 8. We note that (4.13) is equivalent to $D(-z_0) \sim e^{\frac{1}{4}\pi i} z_0^{-\frac{1}{2}}$.

5. The solution for a linear velocity profile†

This model for the primary flow is exceptional in that the equation for $F(\eta)$ is exactly the Orr–Sommerfeld equation, which moreover can be solved completely in this case: consequently, it is worth brief consideration as an example on which to test some of the foregoing theory. The model obviously suffers from lack of realism, its practical applications being essentially restricted to cases where the disturbance of the flow is confined to a region over which a linear approximation can be made to the actual velocity profile. Since the disturbance penetrates to distances of the order of a wavelength from the boundary, the model would generally be reliable only for very short waves.

When $U(\eta) = G\eta$, where G is a constant (see figure 4(*a*)), the exact equation (3.1) for $F(\eta)$ becomes

$$im^3(\eta - c/G)(F'' - k^2F) = F^{\text{iv}} - 2k^2F'' + k^4F, \tag{5.1}$$

with $m^3 = kRG$. The complete solution of (5.1) which tends to zero for large η is $F = \phi + f$, where

$$\phi = A\,e^{-k\eta}, \tag{5.2}$$

and

$$f = B\int_C \exp\left[\tfrac{1}{3}t^3 + \{im(\eta - c/G) + \alpha^2\}t\right]\frac{dt}{t^2+\alpha^2}, \tag{5.3}$$

in which $\alpha = k/m$, and the path of integration is the same as in (4.2). Again C must pass over the origin and be terminated at $-\infty$ and $\infty\, e^{\frac{1}{3}\pi i}$, but is otherwise arbitrary. Although it appears that the integral (5.3) takes different values accordingly as C passes above or below the singularity at $t = i\alpha$, this choice of path is in fact immaterial, since the residue from $t = i\alpha$ simply reproduces the solution (5.2). The mathematical steps leading to (5.3) are not difficult, and an account of them may reasonably be omitted: it is easily verified by differentiation that (5.3) is a solution of (5.1). The present ϕ and f have precisely the meanings associated with these symbols in § 3, but are exceptional in that both are exact solutions.

It is not possible to express the required quantities $f(0)$, $f'(0)$, etc., exactly in forms any simpler than the integrals given directly by (5.3). To obtain compact results, therefore, approximate expressions will have to be introduced. This step nullifies to an extent the advantage of having exact solutions, which is the outstanding merit of the present model; but there remains the advantage that the scheme of approximation can be extended without difficulty to successively higher degrees of accuracy.

As α may be assumed to be small, which corresponds to assumption (ii) of (2.18), f may suitably be approximated in terms of ascending powers of α. The factor $\exp(\alpha^2 t)$ in the integrand of (5.3) can be expanded as a power series and

† I have recently learnt that this example was also treated by Dr M. S. Longuet-Higgins in unpublished work done in 1952 while he was at the Scripps Institute of Oceanography, La Jolla. By means of a theory developed in Cartesian co-ordinates, he calculated the pressure distribution over a solid corrugated surface bounding a uniform shearing flow. His results agree with the present ones for a solid boundary.

integrated term by term, since this expansion is convergent for all t and the series of integrals is also convergent. Now, the binomial expansion

$$\frac{1}{t^2+\alpha^2} = \frac{1}{t^2} - \frac{\alpha^2}{t^4} + \frac{\alpha^4}{t^6} - + \ldots$$

is convergent for $|t| > \alpha$, and so this expansion can clearly be used in the integrand and integrated term by term if the path C passes outside the circle of non-convergence $|t| = \alpha$. But the integrand of each integral in this expansion has no finite singularity other than at $t = 0$, so that by Cauchy's theorem each integral along C outside $|t| = \alpha$ is the same as that along the path used in § 4, i.e. along the two radii and small arc shown in figure 2. Thus (5.3) can be evaluated by expanding the integrand as suggested and integrated term by term along the path in figure 2. For example, we would obtain in this way, with $c = 0$,

$$f(0) = B\{I_0 + \alpha^2(I_1 - I_{-2}) + \alpha^4(I_2 + I_{-4}) + \ldots\},$$

where I_N is the function of N ($= 0, \pm 1, \pm 2, \ldots$) defined below (4.10). Hence, the problem can in principle be completed to any order of approximation in terms of α. However, it seems enough for present purposes to take only the leading term in the expansion of (5.3), which is the function defined by (4.2). That is, we neglect $O(\alpha^2)$. Terms which are $O(\alpha)$ will arise through the boundary conditions; and these seem sufficient to illustrate the effect of the parameter α.

Owing to the fact that the critical point $\eta_c = c/G$ is not a singular point in this example, the various cases considered separately in § 4 can conveniently be covered at once by taking the form (4.12) of the boundary condition and allowing c to have any value including zero. If $f'(0)/f(0)$ is expressed in terms of the Tietjens function according to (4.3), and (5.2) is used for $\phi(0)$ and $\phi'(0)$, this boundary condition gives

$$A - c = -m^{-1}D(-z_0)\{-kA + G\}$$

with $z_0 = mc/G$, and hence

$$A = -Gm^{-1}\{D(-z_0) - z_0\}\{1 + \alpha D(-z_0)\} \tag{5.4}$$

to the first order in α.

We now find the normal stress σ_s on the boundary, which (2.13) shows to be the same as the pressure $p_s = \mathscr{R}\{P_s e^{ik\xi}\}$ to the present order of approximation. Equation (3.9) gives with the same accuracy simply $P_s = (G - kc)A$; and so, using (5.4), we have

$$P_s = -k^{-\frac{1}{3}}R^{-\frac{1}{3}}G^{\frac{5}{3}}(1 - \alpha z_0)\{D(-z_0) - z_0\}\{1 - \alpha D(-z_0)\}. \tag{5.5}$$

This relation shows both the real and imaginary parts of P_s to vary in a very complicated way with the parameter z_0. Consider first the real part $(P_s)_r$, which measures the amplitude of the pressure component in phase with the wave elevation. If we ignore the quantity $\alpha D(-z_0)$, which is always small since $D(-z_0)$ is never large, (5.5) shows $(P_s)_r$ to be negative—as it is for a potential flow over waves with U constant—when $(1 - \alpha z_0)\{D_r - z_0\} > 0$. This condition is satisfied only with $z_0 < 0{\cdot}9$ and $z_0 > \alpha^{-1}$: that is, with $c < 0{\cdot}9G/m$ and $c > G/k$. The imaginary part $(P_s)_i$ is positive, which implies a positive 'sheltering coefficient'—whose significance is to be explained later, when $(1-\alpha z_0)D_i < 0$. This occurs with $z_0 < 2{\cdot}3$ and $z_0 > \alpha^{-1}$. The occurrence of positive values of $(P_s)_i$ for a range of small values of c and then only for fairly large values will also be observed in § 7, where velocity profiles more realistic than the present one will be treated. This will be pointed out to have considerable physical interest.

An explicit expression for p_s will now be written down with $c = 0$, this case being sufficient for purpose of illustration. According to (4.6), we have $D(0) = 0{\cdot}776\, e^{-\frac{1}{6}\pi i}$. Hence, (5.5) leads immediately to†

$$p_s = -0{\cdot}776ak^{-\frac{1}{3}}R^{-\frac{1}{3}}G^{\frac{2}{3}}\{\cos(k\xi-\tfrac{1}{6}\pi)+0{\cdot}776\alpha\cos(k\xi-\tfrac{1}{3}\pi)\}. \qquad (5.6)$$

The shearing stress τ_s on the wave surface can also be calculated easily for this case. Equation (2.12) gives

$$\begin{aligned} T_s &= R^{-1}\{f''(0)+k^2A\} \\ &= R^{-1}A\{-f''(0)/f(0)+k^2\}. \end{aligned} \qquad (5.7)$$

Using (5.4) together with the expression (4.7) for $f''(0)/f(0)$, and neglecting terms which are $O(\alpha^2)$, we obtain directly

$$T_s = 1{\cdot}065\, e^{\frac{1}{6}\pi i}R^{-1}mG(1+0{\cdot}776\, e^{-\frac{1}{6}\pi i}\alpha). \qquad (5.8)$$

The stress τ_s is the real part of $T_s a\, e^{ik\xi}$: thus, when we put $m = (kRG)^{\frac{1}{3}}$, (5.8) shows that

$$\tau_s = 1{\cdot}065aR^{-\frac{2}{3}}k^{\frac{1}{3}}G^{\frac{4}{3}}\{\cos(k\xi+\tfrac{1}{6}\pi)+0{\cdot}776\alpha\cos k\xi\}. \qquad (5.9)$$

Note that the amplitude of the first cosine term may be written $1{\cdot}065\tau_0 ma$, where $\tau_0 = G/R$ is the uniform shearing stress exerted by the primary flow.

In these formulae x may be written instead of ξ if preferred. For comparison with them it is appropriate to recall from (2.1) that the wave surface is expressed in terms of (x, y) by the equation $y = a\cos kx$. Note that both τ_s and σ_s vanish for $R \to \infty$.

An interesting feature of (5.6) and (5.9) is the phase relation between the stresses and the wave on the boundary. It is seen that at high Reynolds numbers ($\alpha \to 0$) the shearing stress is approximately $\frac{1}{6}\pi$ in advance of the wave: that is, the maxima occur at points one-twelfth of a wavelength in the rear of the wave crests (x being regarded as the forward direction on the waves). The phase of the normal stress is $\frac{5}{6}\pi$ in advance: that is, the maxima occur one-twelfth of a wavelength forward from the troughs. These results, particularly that for the phase of τ_s, can be interpreted to indicate a kind of 'sheltering' on the leeward slopes of the waves; but this will be made much clearer by an example which is to be considered as part of § 8. At least it can be said that, despite the rather artificial character of this linear-profile model, these phase relationships seem perfectly in keeping with intuition about the general nature of wind forces on waves (we recall the useful remarks by Lamb (1932, p. 630) concerning what is to be expected in this matter).

† This formula was also obtained by Dr Longuet-Higgins in his work cited in the previous footnote.

6. A simple profile approximating a laminar boundary layer

As a further simple example, we consider a velocity profile characterized by a region $0 \leqslant \eta \leqslant \delta$ (region (i)) in which U is given by

$$U = U_\infty \sin K\eta \qquad (K = \pi/2\delta), \tag{6.1}$$

and beyond which (in region (ii)) U remains constant at the value U_∞ (see figure 4(*b*)). This profile is a fairly close approximation to the laminar boundary layer along a plane, if we take $\delta = 4{\cdot}8XR_X^{-\frac{1}{2}}$, X being the distance from the start of the boundary layer and R_X the Reynolds number based on this distance (cf.

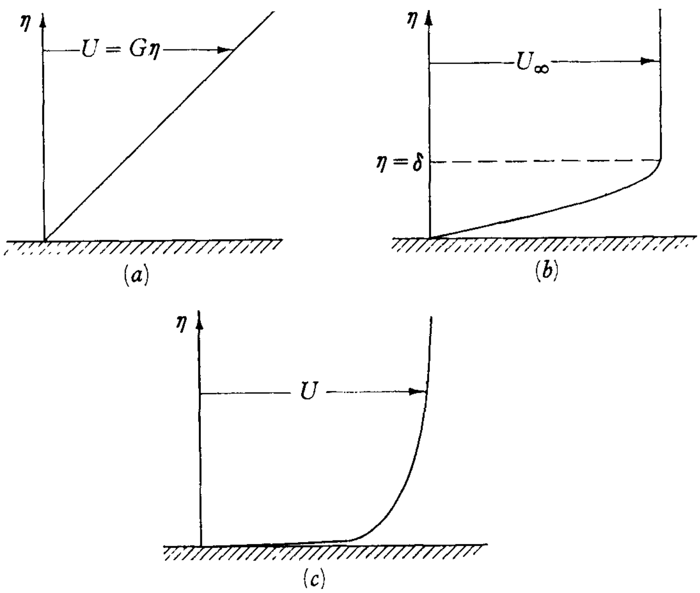

FIGURE 4. Types of velocity profile considered: (*a*) linear profile; (*b*) laminar boundary-layer profile approximated by a quarter-period sinusoid; (*c*) turbulent boundary-layer profile.

Lamb 1932, p. 686). With this value of δ, the important parameter $U'(0) = KU_\infty$ is the same as for the exact boundary-layer profile (see, for instance, Schlichting 1955, chapter VII); and it is clear that the small differences between the present profile and the exact one would scarcely affect our analysis. This profile is specially worth considering for the case $c = 0$, since the differential equation (3.3) for ϕ can be solved exactly.

When (5.1) is substituted, equation (3.3) becomes, with $c = 0$,

$$\phi'' + (K^2 - k^2)\phi = 0, \tag{6.2}$$

whose general solution may be expressed

$$\phi = U_\infty(A \sin l\eta + B \cos l\eta), \tag{6.3}$$

where $l = \sqrt{(K^2 - k^2)}$. It also appears from (3.3) that the form of the inviscid solution in region (ii) is simply $\phi = \text{const.} \times e^{-k\eta}$. Therefore, to make ϕ and ϕ'

continuous at $\eta = \delta$, the solution (6.3) must satisfy the condition $\phi'(\delta) = -k\phi(\delta)$. Hence, putting $\theta = l\delta = \frac{1}{2}\pi\sqrt{\{1-(k/K)^2\}}$, we get

$$\frac{A\cos\theta - B\sin\theta}{A\sin\theta + B\cos\theta} = -\frac{k}{l}, \tag{6.4}$$

which gives on rearrangement

$$\frac{A}{B} = \frac{1-(k/l)\cot\theta}{\cot\theta + (k/l)}. \tag{6.5}$$

For use in the boundary condition (4.8), we have $\phi(0) = BU_\infty$, $\phi'(0) = lAU_\infty$ and $U'(0) = KU_\infty$. Thus, if (6.5) is used to eliminate A, (3.7) leads to

$$B\left\{\frac{l-k\cot\theta}{\cot\theta+(k/l)} + 1{\cdot}288\, e^{\frac{1}{6}\pi i} m\right\} = -K. \tag{6.6}$$

This relation determines B, and hence A is known through (6.5).

To find the pressure at the boundary, we may use the formula (3.12), putting $\epsilon = 0$ as we can do if the minute pressure variation across the wall layer is to be neglected. This gives

$$P_s = U'(0)\,\phi(0) = KBU_\infty^2 = -\frac{K^2U_\infty^2}{l}\left\{\frac{1-(k/l)\cot\theta}{\cot\theta+(k/l)} + 1{\cdot}288\, e^{\frac{1}{6}\pi i}\frac{m}{l}\right\}^{-1}. \tag{6.7}$$

Following from (2.12), an approximation to the shearing stress on the boundary is provided by

$$T_s = R^{-1}f''(0) = -R^{-1}\{f''(0)/f(0)\}\,\phi(0). \tag{6.8}$$

Hence, using (4.7) and (6.6), and noting that in the present example

$$m = [kKRU_\infty]^{\frac{1}{3}},$$

we find that

$$T_s = 1{\cdot}372\, e^{\frac{1}{3}\pi i}\frac{K^2kU_\infty^2}{lm}\left\{\frac{1-(k/l)\cot\theta}{\cot\theta+(k/l)} + 1{\cdot}288\, e^{\frac{1}{6}\pi i}\frac{m}{l}\right\}^{-1} \tag{6.9}$$

These expressions for the stresses are too complicated for easy interpretation; but considerable simplification is possible if the boundary-layer thickness δ is taken to be small compared with the wavelength, so that $k\delta$ and hence k/K are small—though still $k\delta \gg R^{-1}$. We need to assume that the wall friction layer still occupies only a small part of the boundary layer; and we also assume that $\Delta = km/K^2$ is small. The checking of these two assumptions will be left until the next section, where a general type of long-wavelength approximation is to be developed. If terms which are $O(k\delta)^2$ are neglected, we have $l \doteqdot K$, $\cot\theta \doteqdot 0$; and (6.7) and (6.9) give approximately

$$P_s = -kU_\infty^2(1 - 1{\cdot}288\, e^{\frac{1}{6}\pi i}\Delta), \tag{6.10}$$

$$T_s = 1{\cdot}372\, e^{\frac{1}{3}\pi i}\, U_\infty^2\, k^2/m. \tag{6.11}$$

The error in (6.10) is $O(\Delta^2)$ and that in (6.11) is $O(\Delta)$, the latter estimate being quite adequate since $T_s \ll P_s$.

These results provide the following explicit expressions for the stresses on the wavy boundary $y = a\cos kx$:

$$p_s/U_\infty^2 = -ka\{\cos k\xi - 1{\cdot}288\Delta\cos(k\xi + \tfrac{1}{6}\pi)\}$$
$$= -ka\{[1 - 1{\cdot}115\Delta]\cos k\xi + 0{\cdot}644\Delta\sin k\xi\}; \tag{6.12}$$

$$\tau_s/U_\infty^2 = 1{\cdot}372ka\alpha\cos(k\xi + \tfrac{1}{3}\pi i), \tag{6.13}$$

where $\alpha = k/m$ as in § 5, this quantity being small by assumption (ii) of (2.18). Note that $\sigma_s = p_s$ to the present order of approximation.

In common with the results found for the previous example, both stresses have distributions suggestive of a 'sheltering' action: that is, their minima occur on the leeward slopes of the waves. The pressure sheltering coefficient according to the usual definition (Jeffreys 1924, 1925; Ursell 1956) is in fact $s = 0{\cdot}644\Delta$.

7. Boundary-layer profiles in general

We shall now investigate some approximate methods applicable generally to velocity profiles of boundary-layer type. We particularly wish, with turbulent boundary layers in mind, to be able to deal with comparatively diffuse profiles of the general form indicated in figure 4 (*c*). Further, it is now time to complete some calculations for the surface stresses in the interesting case where $c > 0$ so that there is a critical point away from the boundary in a region where the profile is curved. To account for the viscous solution $f(\eta)$, the method applied previously is still useful; but it remains to find a suitable approximation to the solution $\phi(\eta)$ of the 'inviscid' equation (3.3). This equation cannot be solved exactly in terms of known functions except in the two simple cases considered previously—and perhaps in one or two other cases of no interest here. If a high degree of accuracy were required, resort would have to be made to numerical solutions computed for particular profiles—as has been done in various studies of hydrodynamic stability (see, for instance, Lin (1945, Part 3) for calculations relating to parabolic and Blasius profiles). It is more instructive, however, to use the approximate expression derived as follows, which is unrestricted to any particular form of U. Although this is essentially a small-k approximation, thus applying most accurately to thin boundary layers, it probably gives a reasonably good account of the properties of ϕ even for fairly diffuse profiles.

We first observe that over a range excluding any critical point an approximation to the solution of (3.3) for small k is

$$\phi = A(U-c)\,e^{-k\eta}, \tag{7.1}$$

which evidently satisfies (3.3) accurately wherever $U''/k^2(U-c)$ is either very small or very large compared with unity. (It appears, rather remarkably, that this approximation was first noted both by Lighthill (1957, § 6) and by Miles (1957, § 5) in papers dealing with entirely different topics but appearing at the same time in the same journal.) Let us for a moment consider the exceptional case where $c < 0$: that is, the wave travels in the direction opposite to the flow. There is now no critical point, and (7.1) is a uniformly valid approximation over the whole of a flow of boundary-layer type. In the region far from the boundary where U tends to a constant value, ϕ tends to const. $\times\, e^{-k\eta}$ as it should do; whereas in the region near the boundary over which U varies rapidly, $\phi \doteqdot$ const. $\times\,(U-c)$ which satisfies (3.3) approximately for small k. Further, if the arbitrary constant A is put equal to -1, the boundary condition (3.7) is satisfied very nearly provided only that the usual conditions $U'(0) \gg kc$ and $U'(0) \gg \beta$ hold; and also (3.5) gives $f = 0$. Hence, (2.4) and (2.5) reduce to $\psi(\xi,\eta) = \psi_0(\eta)$ and $v = 0$, $u = U(\eta)-c$. The ξ-lines are thus *streamlines* according to this approximation.

The latter result incidentally has considerable interest as evidence supporting the method of linearization set out in § 2. For it shows that, at least in the absence of a critical point, a fair approximation to the actual flow pattern is already obtained simply by 'bending' the primary profile to follow the waves by use of the system of curvilinear co-ordinates. There can therefore be no further doubt about the smallness of the periodic perturbation introduced on the left-hand side of (2.4) to represent the difference between the 'bent' profile and the actual flow.

While serving best for $c < 0$, this approximation also has a somewhat limited usefulness in the case $c = 0$, provided $U''(0) = 0$ so that ϕ does not exhibit singular behaviour at the critical point on the boundary. The wall friction-layer is effectively absent in this approximation, and so the surface shearing stress is indeterminate. According to (3.9), however, an approximation to the pressure at the wave surface is given by

$$P_s = -k^2 \int_0^\infty U^2 e^{-k\eta}\, d\eta. \tag{7.2}$$

Thus the pressure is in exact anti-phase with the wave elevation, no sheltering being indicated.

To deal adequately with the case of a rigid boundary, and to make any progress at all with the case where there is a critical point inside the fluid, an improvement on (7.1) is obviously needed. A method of developing successive uniformly valid approximations to the solution of (3.3) for small k was given by Lighthill (1957, § 6). He applied the method under the restriction that $U-c$ (equivalent to his V) does not vanish anywhere; but it still formally yields results in the absence of this restriction; and certain questions of indeterminacy which then arise can be answered by other considerations (see below). A second approximation to the solution of (3.3) is found by Lighthill's method to be

$$\phi = A(U-c)\,e^{-k\eta}\left[1 + k\int_\eta^\infty \left\{\left(\frac{U_\infty - c}{U-c}\right)^2 - 1\right\} d\eta\right], \tag{7.3}$$

where U_∞ is the constant value approached by U as $\eta \to \infty$. The error in this is proportional to k^2. Details of the derivation may be omitted, as it is easily verified that (7.3) satisfies (3.3) to the order of approximation in question.

This approximate expression for ϕ seems quite adequate for our present purpose. Although the integral in (7.3) is indeterminate for $\eta < \eta_c$ (i.e. $U < c$), we shall see that a simple adjustment indicated by the theory of the full Orr–Sommerfeld equation is sufficient to remove the ambiguity, and so make (7.3) give a correct account of ϕ either side of the critical point. It can be shown that the extra terms present in higher approximations (the next one was given by Lighthill) do not affect the behaviour at the critical point in any vital way.

It is noteworthy that (7.3) reduces as follows in the case of very thin boundary layers: i.e. where the boundary-layer thickness δ is very small compared with k^{-1} and we have $U = U_\infty$ for $\eta \geqslant \delta$. If terms which are $O(k\delta)^2$ are neglected, (7.3) gives, for $\eta < \delta$,

$$\phi = A(U-c)\left[1 + k(U_\infty - c)^2 \int_\eta^\delta \frac{d\eta}{(U-c)^2}\right]. \tag{7.4}$$

This expression is seen to be the sum of two functions of η which each satisfy the equation $(U-c)\phi''-U''\phi = 0$ (i.e. equation (3.3) with the term in k^2 omitted); and also (6.4) satisfies the boundary condition which, as demonstrated in § 6, is to be applied at the edge of the boundary layer, i.e. $\phi'(\delta) = -k\phi(\delta)$. It may be recognized that the two functions comprise the leading terms in the well-known Heisenberg expansion (see Lin 1955, p. 34), which expresses the solution of (3.3) in powers of k^2 and has had important applications to stability theory. Unlike (6.3), an approximation taking a few terms of the Heisenberg expansion is not uniformly valid over the whole range $0 \leqslant \eta < \infty$, and would be unsuitable for the application we intend to make to fairly diffuse boundary-layer profiles.

The case of a rigid solid boundary

In our first application of (7.3), let us take $c = 0$ and assume no primary pressure gradient so that $U''(0) = 0$. For clarity in what follows, it seems worth while to rewrite (7.3) with $c = 0$:

$$\phi = AU e^{-k\eta}\left[1+k\int_\eta^\infty \left\{\frac{U_\infty^2}{U^2}-1\right\}d\eta\right]. \tag{7.5}$$

The integral in (6.5) diverges without bound for $\eta \to 0$ (i.e. $U \to \eta U'(0) \to 0$); but its product with U converges to a finite limit. Thus, we easily find that

$$\phi(0) = AkU_\infty^2/U'(0). \tag{7.6}$$

It is also easy to show that $\phi'(0)$ is expressible in the form

$$\phi'(0) = AU'(0)\{1+\chi\}, \tag{7.7}$$

where

$$\chi = \int_0^\infty \left\{1-\frac{U'}{U'(0)}\right\}\left\{\frac{U_\infty^2}{U^2}-1\right\}d(k\eta)-\frac{2kU_\infty}{U'(0)}. \tag{7.8}$$

The integral which χ denotes is clearly convergent to a finite limit if $U''(0) = 0$, as is assumed; for then $\{U'(0)-U'(\eta)\} \to O(\eta^2)$ as $\eta \to 0$, and so the integrand remains finite at $\eta = 0$. If $U''(0)$ were not zero, a logarithmic singularity would arise, as one expects since $\eta = 0$ would in this case be a singular point of the differential equation (3.3). To deal with the case of non-zero pressure gradient, the 'modified inviscid' solutions studied by Tollmien (1929) would be required: these satisfy (3.3) away from the critical point, but in its immediate neighbourhood become approximate solutions of the full Orr–Sommerfeld equation. However, we shall not attempt to cover this case.

If the ratio $kU_\infty/U'(0)$ is specified (or $k\delta$), the quantity in (7.7) depends only on the shape of the velocity profile. For example, we find that $\chi = -k\delta = -\frac{1}{2}\pi kU_\infty/U'(0)$ for the sinusoidal profile considered in § 6. Again, for the exact laminar boundary layer along a plane, a numerical integration using Howarth's figures (Schlichting 1955, p. 107) gives $\chi = -1{\cdot}75kU_\infty/U'(0)$. To estimate χ for turbulent boundary-layer profiles is more difficult, since the greatest contribution to the integral (7.8) comes from the region near the boundary —yet outside the viscous sublayer—where the shape of the profile is least accurately known. However, it is still clear that $\chi = O(k\delta^*)$, say, where δ^* is roughly speaking the width of the region over which most of the variation of

U occurs. This may be considerably less than $O(k\delta)$, where δ is the 'overall' boundary-layer thickness; because, for a turbulent profile, U is not much different from U_∞ over most of its width, and so the respective contribution to (7.8) is small. It is inappropriate to go into this matter any further; for we shall regard χ as a small quantity anyway, and presently see that it may be neglected entirely in forming results which suffice as a first approximation.

If we also take $\Delta = kmU_\infty^2/[U'(0)]^2$ to be small, the substitution of (7.6) and (7.7) in the boundary condition (4.8) gives approximately

$$A = -1\{1-\chi-1{\cdot}288\,e^{\frac{1}{6}\pi i}\Delta\}. \tag{7.9}$$

Hence, if this is put into (7.5) and the integral in (7.5) is combined with the definition (7.8) of χ, the result can be written

$$\phi = -\frac{kU_\infty^2}{U'(0)} - U\,e^{-k\eta}\left[1-1{\cdot}288\,e^{\frac{1}{6}\pi i}\,\Delta - k\int_0^\eta\left\{1-\frac{U'}{U'(0)}\right\}\left\{\frac{U_\infty^2}{U^2}-1\right\}d\eta+\frac{kU}{U'(0)}\right]. \tag{7.10}$$

To find the pressure at the boundary, the integral formula (3.9) is clearly the most appropriate at present, since it is not unduly sensitive to errors in our expression for ϕ. (Note that the alternative formula (3.12) would give a very poor approximation here: it was used without disadvantage in the previous example (§ 6) only because ϕ was known exactly there.) In evaluating (3.9), there is justification for omitting some of the terms in (7.10). As the most important simplification, the integral in (7.10) can be neglected: since the fixed limit in this integral is zero (not ∞ as in (7.5)), its effect tends to be cancelled by the exponential factor multiplying it; and its omission is well justified if $k\delta^*$ is fairly small. Further, two terms which are $O(kU_\infty/U'(0))$ times the leading terms can be neglected, with obvious justification if a turbulent boundary layer is in question, and also if $k\delta$ is small in the case of a laminar boundary layer. Accordingly, we obtain the approximation

$$P_s = -k^2(1-1{\cdot}288\,e^{\frac{1}{6}\pi i}\Delta)\int_0^\infty U^2 e^{-k\eta}\,d\eta. \tag{7.11}$$

The shearing stress can be found in exactly the same way as in § 6. The result to a first approximation is

$$T_s = 1{\cdot}372\,e^{\frac{1}{6}\pi i}\,U_\infty^2\,k\alpha, \tag{7.12}$$

which is the same as (6.11) except that here no specific profile is implied and we have $\alpha = k/m = k^{\frac{2}{3}}R^{-\frac{1}{3}}[U'(0)]^{-\frac{1}{3}}$.

These results probably give reasonable estimates of P_s and T_s even for profiles which are fairly diffuse measured against a wavelength; but clearly they are most accurate when $k\delta$ is small. A survey of the various approximations which have been introduced shows (7.11) and (7.12) to become *exact* as $k\delta \to 0$, provided $m\delta$ still remains small, i.e. provided the assumption that the wall layer takes up only a small part of the boundary layer remains valid for thin boundary layers. This assumption, together with the one that Δ is small, will be examined presently. Note that when $k\delta$ is very small, $U = U_\infty$ over most of the range of integration of the integral in (7.11); and so this equation reduces approximately to (6.10). For very thin boundary layers, therefore, we have that the magnitude *and phase* of both the surface pressure and shearing stress become independent of the profile shape.

We observe from (7.11) that the amplitudes of the component of p_s in phase with the wave elevation and of the component in phase with the slope are respectively given, as fractions of U^2_∞, by

$$a\mathscr{R}\{P_s\}/U^2_\infty = -ka(1-1{\cdot}115\Delta)I, \tag{7.13}$$

$$a\mathscr{I}\{P_s\}/U^2_\infty = 0{\cdot}644ka\Delta I = kas, \tag{7.14}$$

where
$$I = \int_0^\infty (U/U_\infty)^2 e^{-k\eta}\, d(k\eta). \tag{7.15}$$

In (7.14), s is the sheltering coefficient as usually defined. The integral I is easily computed for particular velocity profiles, its value being insensitive to fine details of the profile such as the rather uncertain region near the wall for a turbulent boundary layer.

Consider, for instance, the case where the $\frac{1}{7}$-power law of velocity distribution is a suitable approximation: here the profile is given by $U/U_\infty = (\eta/\delta)^{\frac{1}{7}}$ over most of the region $0 \leqslant \eta \leqslant \delta$, but is rounded off to make the slope continuous at $\eta = \delta$, and is joined smoothly to the straight line $U = \eta U'(0)$ in the viscous sublayer next to the boundary (we recall this is not to be confused with our 'wall layer', which needs to be rather thinner than the sublayer for the present calculations to apply—see § 8). The latter details scarcely affect I, to which therefore a good approximation is

$$I = (k\delta)^{-\frac{2}{7}} \int_0^{k\delta} z^{\frac{2}{7}} e^{-z}\, dz + \int_{k\delta}^\infty e^{-z}\, dz.$$

The first integral may be expressed in terms of the incomplete gamma function. Successive integration by parts gives the following asymptotic expansion for large $k\delta$:

$$I = \frac{\Gamma(\frac{9}{7})}{(k\delta)^{\frac{2}{7}}} - e^{-k\delta}\left\{\frac{2}{7k\delta} - \frac{10}{49(k\delta)^2} + \ldots\right\}. \tag{7.16}$$

It may also be of interest to calculate I for the 'universal' logarithmic law of velocity distribution for turbulent flow over a smooth plane (Schlichting 1955, p. 406). This may be expressed

$$U/U_* = 2{\cdot}5 \log (9U_* R\eta), \tag{7.17}$$

where U_* is Prandtl's friction velocity defined by $U^2_* = \tau_0 = R^{-1}U'(0)$: also $U^2_*/U^2_\infty = c_f$, the usual 'local coefficient of skin friction'. Hence, (7.15) gives

$$I = 6{\cdot}3c_f \int_0^\infty [\log (9U_* R\eta)]^2 e^{-k\eta}\, d(k\eta).$$

This integral is identifiable with one of the Laplace transforms in the tables edited by Erdélyi (1954, § 4.6, no. 13). We find that

$$I = 6{\cdot}3c_f\{\tfrac{1}{6}\pi^2 + [\log (\gamma k/9U_* R)]^2\}, \tag{7.18}$$

where $\gamma = e^C$, $C \doteqdot 0{\cdot}5772$ being Euler's constant.

A comparison with Motzfeld's experiments

The measurements by Motzfeld (1937) of the pressure distribution over a rigid wavy surface placed in a wind tunnel were mentioned in § 1. For his first model the wave amplitude and length were 0·75 and 30 cm, so that $ka = 2\pi a/\lambda = 0{\cdot}157$; and the observed pressure distribution was very nearly sinusoidal, with its maximum slightly in advance of the wave trough, as predicted by the theory. We shall now compare his measurements of the in-phase pressure amplitude and sheltering coefficient with values given by the preceding formulae. Remembering that Motzfeld's wave-train was only three wavelengths long (a feature criticized adversely by Ursell (1956)) and that there was a considerable pressure gradient along the tunnel, it will be seen that the agreement is as good as one could reasonably expect.

Motzfeld estimated values of the skin-friction coefficient c_f by fitting the universal logarithmic curve to the observed velocity profile in the wind tunnel. At the Reynolds number of the tests on his first wave model, he found $c_f = 0{\cdot}00173$. Now it is easily shown that Δ is equivalent to $c_f^{-\frac{1}{3}} R_\lambda^{-\frac{2}{3}}$, where R_λ is the Reynolds number based on U_∞ and wavelength (i.e. $R_\lambda = U_\infty \lambda/\nu$ if U_∞ and λ express *dimensional* quantities). If U_∞ is identified with the maximum air velocity (at the middle of the tunnel), we have $R_\lambda = 330{,}000$, and so obtain $\Delta = 0{\cdot}0021$.

The presence of the opposite wall of the tunnel evidently had an insignificant effect on conditions at the wave surface. This is shown by Motzfeld's measurements of the velocity profile at different positions over a wavelength: it is also indicated by the fact that $k\eta = 4{\cdot}2$ at the opposite wall, so that $e^{-k\eta}$ was quite small there. To find I, the most satisfactory course is to make a numerical integration based on the measured profile. This gives $I = 0{\cdot}71$. (We remark that $k \times$ (*half the tunnel height*) is rather too large for the formulae (7.16) and (7.18) to be reliable; and in fact they considerably overestimate I.)

Hence, we estimate the right-hand side of (7.13) to be $-0{\cdot}11$. The corresponding experimental result is $-0{\cdot}10$. Our estimate of the sheltering coefficient is $s = 0{\cdot}010$; whereas the experimental value is 0·034 (note Motzfeld's c_d, values of which are given in his table 1, is equivalent to $(ka)^2 s$). The discrepancy between these values of s may well be attributable to the experimental difficulties of measuring the very small pressure component in phase with the wave slope: the amplitude of this component was only a few percent of the amplitude of the total distribution.

Concluding remarks on the analysis for a rigid boundary

It remains to check the general assumption of small Δ and also the assumption that $m\delta$ remains small for limitingly thin boundary layers. We shall incidentally consider how our formulae for the surface stresses reduce when the Reynolds number is made infinite.

It is suitable just here to relax the scheme of using dimensionless variables and assign dimensions appropriately to all symbols. The only change of notation required by this is to replace R by ν^{-1} everywhere: thus, for instance, we now have $m = [kU'(0)/\nu]^{\frac{1}{3}}$, which has the dimension of (length)$^{-1}$. The parameters Δ and $m\delta$ to be considered are, of course, still dimensionless.

Consider first a laminar boundary layer, formed in the absence of pressure gradient, at a distance X from its starting point. To comply with the basic assumption that variations in the primary profile are negligible over a wavelength, X must be fairly large in comparison with the wavelength λ: that is, kX must be large compared with unity. This requirement tends to conflict with one that $k\delta$ should be small; but evidently both can be satisfied if the Reynolds number R_X defined below is sufficiently large. The familiar Blasius formula gives $U'(0) = 0{\cdot}332\nu^{-\frac{1}{2}}X^{-\frac{1}{2}}U_\infty^{\frac{3}{2}}$; and a fair estimate of the boundary-layer thickness is $\delta = 6\nu^{\frac{1}{2}}X^{\frac{1}{2}}U_\infty^{-\frac{1}{2}}$. Hence, there easily follows

$$m\delta = 4{\cdot}2(kX)^{\frac{1}{6}}, \tag{7.19}$$

and

$$\Delta = kmU_\infty^2/[U'(0)]^2 = 6{\cdot}3(kX)^{\frac{4}{3}}R_X^{-\frac{1}{2}}, \tag{7.20}$$

where $R_X = U_\infty X/\nu$. It is also found that

$$\alpha = 0{\cdot}230\Delta/(kX)^{\frac{2}{3}}. \tag{7.21}$$

Equation (7.19) shows that $m\delta \gg 1$ as required; and (7.20) shows that, with X fixed, Δ becomes small when R_X is made sufficiently large (e.g. by increasing the velocity U_∞ in the main stream). Clearly, the justification for assuming Δ small is not in this example as readily forthcoming as one might wish; but we remark that anyway it would be no great trouble to calculate expressions for the stresses without restriction on Δ if this were thought worth while.

By means of (7.21) a clear comparison can be made between the magnitudes of the pressure component in phase with the wave slope and of the shear-stress component in phase with the wave elevation. This comparison has interest in the following way. Suppose the wave to be travelling forward *slowly*, so that although $c > 0$ the present calculations still apply approximately, i.e. we have the second special case considered in § 4. Then each of these stress components does a proportional amount of work on the wave, the respective rates of energy transfer being the same when the stresses have the same magnitude (cf. Lamb 1932, § 350). According to (7.12) and (7.14), the ratio of the shear-stress component to the pressure component is $O(\alpha/\Delta)$, which is seen from (7.21) to be very small and—rather surprisingly—independent of Reynolds number.

In studies of viscous fluid motion there is always interest in what happens in the limit as the Reynolds number is made to tend uniformly to infinity. It is well known that the limiting character of the motion may differ from that according to the theory of inviscid fluids. The present approximations evidently improve with increasing R_X, provided X is fixed; and there is no doubt about their remaining valid as $R_X \to \infty$. In this limit we get $\Delta = \alpha = 0$, and the boundary-layer thickness shrinks to zero, so that (7.11) and (7.12) give $p_s = -kaU_\infty^2 \cos k\xi$ and $\tau_s = 0$. This expression for the pressure is that given by ideal-fluid theory applied to the (Kelvin–Helmholtz) model of a uniform primary flow extending right down to the boundary.

Next consider a turbulent boundary layer with a velocity distribution approximated by the $\frac{1}{7}$-power law. For this form of boundary layer, Schlichting (1955,

p. 433) gave $\delta = 0{\cdot}37XR_X^{-\frac{1}{5}}$ and $U'(0) = 0{\cdot}0296\nu^{-1}U^2R_X^{-\frac{1}{5}}$. These formulae lead to

$$m\delta = 0{\cdot}12(kX)^{\frac{1}{3}}\,R_X^{\frac{2}{5}}, \tag{7.22}$$

$$\Delta = 350(kX)^{\frac{4}{3}}\,R_X^{-1}, \tag{7.23}$$

and

$$\alpha = 0{\cdot}0092\Delta(kX)^{-\frac{2}{3}}\,R_X^{\frac{1}{5}}. \tag{7.24}$$

We recall that the $\frac{1}{7}$-power law is valid in the range $5 \times 10^5 < R_X < 10^7$. With such values of R_X, (7.22) shows the condition $m\delta \gg 1$ to be very amply satisfied; and (7.23) shows we may have quite large values of kX while still having $\Delta \ll 1$. The two requirements are far more readily met than in the previous case. Equation (7.24) indicates that the ratio α/Δ is again small; hence the action of pressure is again more effective than that of shearing stress in supplying energy to a slowly moving wave. The results in the limit as $R_X \to \infty$ are the same as before.

Waves progressing with a fair speed in the direction of flow

We now turn to the case where there is a critical point $U = c$ at some significant distance $\eta = \eta_c$ from the boundary. The first task is to decide the correct interpretation of (7.3) for $\eta < \eta_c$: this expression is at present ambiguous owing to the singularity of the integrand at $\eta = \eta_c$. The difficulty is common to any uniformly valid approximation to the solution of (3.3), and can only be resolved by matching ϕ to the appropriate solutions of the full Orr–Sommerfeld equation which, unlike the inviscid solution, remain valid through the vicinity of the critical point. (This recalls our remarks following equation (3.3).) As this matter has been very fully examined by Lin (1955, chapter 8) and others in the context of stability theory, there is no need to go into it in any detail here. The relevant conclusion is that (7.3) becomes a physically correct approximation on either side of a narrow region surrounding the critical point if, for $\eta < \eta_c$, the path of integration is indented below the real axis under the singularity. The thickness of the 'friction layer' around the critical point is $O(kRU_c')^{-\frac{1}{3}}$ and so may be assumed to be very small. The derivatives of (7.3) cease to be valid approximations in this region. Note, however, that the approximation to $\phi(\eta)$ itself is valid *everywhere*, as (7.3) remains finite at $\eta = \eta_c$: thus it is clearly quite safe to use (7.3) in the integral expression (3.9) for the pressure.

The contribution to the integral in (7.3) from an infinitesimal indentation under the singularity is readily found by the calculus of residues. This contribution will be seen presently to play a crucial part in determining the pressure component in phase with the wave slope; and this result is obviously closely related to the matters discussed at the end of § 3, where we recalled the theory of Miles (1957) showing that a quasi-sheltering action can arise from the effects of a critical point. This is the principal effect due to the integral part of our expression for ϕ, which is otherwise of secondary importance. It appears, therefore, that the usefulness of the present approximation can be greatly improved by the slight modification as follows, which makes (7.3) give an accurate result at the critical point.

We first remark that although the integral in (7.3) is to be regarded as a second-order term for most values of η, it becomes dominant near the critical point. For

$\eta = \eta_c$ exactly, the integral is infinite; but, as was noted in connexion with (7.6) and also two paragraphs ago, the whole expression gives the finite value

$$\phi_c = Ak(U_\infty - c)^2 e^{-k\eta_c}/U'_c. \tag{7.25}$$

Here we write $\phi_c \equiv \phi(\eta_c)$ and $U'_c \equiv U'(\eta_c)$. Now, this will be shown later to be a rather poor approximation to ϕ_c if A is determined by the boundary conditions. A better approximation can be found quite easily by other means, as will be done later; and so a useful course is to rewrite (7.3) in the form

$$\phi = (U-c)\,e^{-k\eta}\left[A + \frac{\phi_c U'_c e^{k\eta_c}}{(U_\infty - c)^2}\int_\eta^\infty \left\{\left(\frac{U_\infty - c}{U-c}\right)^2 - 1\right\}d\eta\right], \tag{7.26}$$

which gives $\phi \to \phi_c$ for $\eta \to \eta_c$, the value of ϕ_c being left undetermined for the present. This expression keeps the same status as (7.3) as a second-order approximation in terms of k, since (7.25) is accurate to a first approximation in k.

A more explicit form of (7.26) for $\eta < \eta_c$ readily follows when the substitution $U-c = U'_c(\eta-\eta_c) + \frac{1}{2}U''_c(\eta-\eta_c)^2 + O(\eta-\eta_c)^3$ is made in the integrand. The singularity is circuited by a small semicircle under the real axis; and hence we are led to the expression

$$\phi = (U-c)\,e^{-k\eta}\left[A - i\pi\phi_c e^{k\eta_c}\frac{U''_c}{U'^2_c} + \frac{\phi_c U'_c e^{k\eta_c}}{(U_\infty - c)^2}\,\mathscr{P}\int_\eta^\infty \left\{\left(\frac{U_\infty - c}{U-c}\right)^2 - 1\right\}\right], \tag{7.27}$$

where $\mathscr{P}$ indicates the principal value of the integral. For $\eta > \eta_c$, this expression is of course to be replaced by (7.26).

The difference between (7.26) and (7.27) is clearly going to make P_s a complex quantity when ϕ is substituted in the formula (3.9). However, we must also anticipate the possibility of quasi-sheltering as a consequence of the exact 'viscous' boundary condition on ϕ, which includes the complex quantity $f'(0)/f(0)$. But, recalling § 4, we expect the latter effect to be significant only if the ratio $mc/U'(0)$ is fairly small and the critical point lies very close to the boundary. Nevertheless, this case definitely seems relevant to turbulent flows, for which the critical point may lie even inside the viscous sublayer for quite large values of the fraction c/U_∞. (It is of interest to note that a conservative estimate of the thickness of the sublayer is indicated by $U/U_* = 5$, U_* being Prandtl's friction velocity. Thus, the fraction of U_∞ attained at the edge of the sublayer is $5U_*/U_\infty = 5c_f^{\frac{1}{2}}$. With the typical value $c_f = 0{\cdot}0025$, this fraction is $\frac{1}{4}$.)

From (7.27) we obtain

$$\phi(0) = -c(A - \Omega), \tag{7.28}$$

where

$$\Omega = i\pi\phi_c e^{k\eta_c}\frac{U''_c}{U'^2_c} - \frac{\phi_c U'_c e^{k\eta_c}}{(U_\infty - c)^2}\,\mathscr{P}\int_0^\infty \left\{\left(\frac{U_\infty - c}{U-c}\right)^2 - 1\right\}d\eta. \tag{7.29}$$

In what follows we take the magnitude of Ω to be reasonably small compared with unity, which implies (as in our treatment of χ earlier) that one wavelength reasonably well covers the region of the profile where most of the variation of U occurs. (Perhaps a fair estimate of the order of magnitude of Ω would be $k \times$ (*momentum thickness of boundary layer*).) Equation (7.27) also leads to

$$\phi'(0) = U'(0)\{A - \Omega\} + \phi_c U'_c e^{k\eta_c}/c, \tag{7.30}$$

where some terms of the order of $kc\Omega$ have been dropped, being negligibly small.

These expressions satisfy the boundary condition (4.12) to a first approximation if $A = -1+\Omega$. To obtain a second approximation, we write $A = -1+\Omega+\epsilon$ and, neglecting squares and products of Ω and ϵ, we obtain from (4.12)

$$\epsilon = -\frac{\phi_c U_c' e^{k\eta_c}}{c^2\left\{\dfrac{f'(0)}{f(0)}+\dfrac{U'(0)}{c}\right\}}, \tag{7.31}$$

which may also be written

$$\epsilon = \frac{\alpha\phi_c U_c' e^{k\eta_c}}{c^2\left\{\dfrac{1}{D(-z_0)}-\dfrac{1}{z_0}\right\}}, \tag{7.32}$$

where we have introduced the Tietjens function D and the notation $\alpha = k/m$, $z_0 = mc/U'(0)$. It should be noted that the approximation to A thus obtained remains valid even when c is so small that the critical point is brought down to the linear region of the profile very close to the boundary; for, though the integral in (7.29) is then $O\{kU^2/U'(0)\,c\}$ and so becomes infinite for $c \to 0$, this order of magnitude is still small if the critical point is not so extremely close to the boundary as to lie well inside the wall friction-layer, i.e. $m\eta_c$ is not $\ll 1$. In fact, for $m\eta_c = 1$, this quantity is equivalent to Δ, which was shown earlier to be small. The quantity ϵ is also of this order when $m\eta_c = O(1)$.

Hence, eliminating A from (7.26) and (7.27), we obtain

$$\phi = (U-c)\,e^{-k\eta}\left[-1+\epsilon+\langle i\pi\phi_c e^{k\eta_c} U_c''/U_c'^2\rangle - \frac{\phi_c U_c' e^{k\eta_c}}{(U_\infty - c)^2}\mathscr{P}\int_0^\eta \left\{\left(\frac{U_\infty - c}{U-c}\right)^2 - 1\right\}d\eta\right], \tag{7.33}$$

where the quantity indicated by $\langle\ \rangle$ is to be put equal to zero for $\eta < \eta_c$. This result is now used in (3.9) to find the pressure at the wave surface. To estimate the pressure component in phase with the wave, it seems justifiable to neglect the small quantity ϵ and the integral in (7.33) (the second simplification may be justified in the same way as the corresponding step made when proceeding from (7.10)). Thus, as an approximation likely to be good enough for most practical applications, the amplitude of this component can be expressed

$$a\mathscr{R}\{P_s\} = -k^2 a\int_0^\infty (U-c)^2 e^{-k\eta}\,d\eta. \tag{7.34}$$

The calculation of the component in phase with the wave slope is more interesting. Writing its amplitude as $(U_\infty - c)^2 kas$ and assuming ϕ_c to be real, we deduce from (7.33) and (3.9) that $s = s_1 + s_2$ where

$$s_1 = \frac{\pi\phi_c k\,e^{k\eta_c} U_c''}{(U_\infty - c)^2 U_c'^2}\int_{\eta_c}^\infty (U-c)^2 e^{-k\eta}\,d\eta, \tag{7.35}$$

$$s_2 = \mathscr{I}\left\{\left(\frac{1}{D(-z_0)}-\frac{1}{z_0}\right)^{-1}\right\}\left[\frac{\phi_c\alpha U_c' e^{k\eta_c}}{c^2(U_\infty - c)^2}\right]\int_0^\infty (U-c)^2 e^{-k\eta}\,d\eta. \tag{7.36}$$

An estimate of ϕ_c is now needed, together with a check that it is real. By integrating (3.3) under the condition $\phi \to 0$ for $\eta \to \infty$, it can be shown (cf. Miles 1957, p. 193) that

$$U'_c \phi_c = k^2 \int_{\eta_c}^{\infty} (U-c)\,\phi\, d\eta. \tag{7.37}$$

(This relationship is exact; and it has in fact already been noted implicitly in the paragraph containing equation (3.12).) A sufficiently accurate approximation to ϕ_c is obtained by taking the leading term of (7.33) and using it for ϕ on the right-hand side of (7.37); thus (7.37) gives

$$\phi_c = -\frac{k^2}{U'_c}\int_{\eta_c}^{\infty} (U-c)^2 e^{-k\eta}\, d\eta, \tag{7.38}$$

which shows ϕ_c to be real in this approximation. This is clearly a much better estimate than the value $-k(U_\infty - c)^2 e^{k\eta_c}/U'_c$ given by (7.25) with $A \doteqdot -1$; but the two become the same for very thin boundary layers. It may be remarked that the accuracy of our treatment of the case $c = 0$ could have been somewhat improved by taking a similar estimate for $\phi(0)$ instead of (7.6); but this scarcely seemed worth while. The only changes in the results for that case are readily seen to be as follows: Δ is to be multiplied by the integral I (e.g. in the formula (7.14) for the sheltering coefficient); and the expression (7.12) for the shearing stress is also to be multiplied by I.

The component sheltering coefficients can now be expressed in reasonably neat forms. Some simplification is allowable if the critical point is assumed to be considerably less than a wavelength away from the boundary, so that $e^{k\eta_c} \doteqdot 1$. This assumption is, of course, consistent with the overall scheme of approximation, provided we exclude the unusual case where c is very nearly equal to U_∞. Hence, there finally are obtained

$$s_1 = -k\pi(U_\infty - c)^2 \frac{U''_c}{U'^3_c} J^2, \tag{7.39}$$

$$s_2 = -\mathscr{I}\left\{\left(\frac{1}{D(-z_0)} - \frac{1}{z_0}\right)\right\} \alpha\left(\frac{U_\infty}{c} - 1\right)^2 J^2, \tag{7.40}$$

where $$J = \int_{\eta_c}^{\infty} \left(\frac{U-c}{U_\infty - c}\right)^2 e^{-k\eta}\, k\, d\eta.$$

The separation of s into these two components is interesting since it illustrates two physically distinct 'modes' of sheltering. First, s_1 represents an effect due entirely to the critical point, and is independent of Reynolds number; however, we emphasize that s_1 is decidedly a real-fluid property, since the singular behaviour of ϕ at the critical point—upon which s_1 crucially depends—remains ambiguous unless the effect of viscosity there is considered. Equation (7.39) clearly corresponds to (3.16), and is also equivalent to the expression for the sheltering coefficient derived by Miles (1957, equation (5.2)) on the basis of an inviscid-fluid model, the ambiguities necessarily entailed by the presence of a critical point being resolved in the same way as here (see p. 192 of his paper). Note that s_1 would usually be positive since U''_c would usually be negative.

On the other hand, s_2 essentially depends on the action of viscosity at the boundary; and, recalling the result found in § 5, we may expect that s_2 will not be positive unless z_0 is fairly small. If z_0 is large enough for the asymptotic formula (4.14) to apply, then (7.40) gives very approximately

$$s_2 = \left(\frac{k}{2Rc}\right)^{\frac{1}{2}} \left(\frac{U_\infty}{c} - 1\right)^2 J^2 = -\pi^{\frac{1}{2}} R_w^{-\frac{1}{2}} \left(\frac{U_\infty}{c} - 1\right)^2 J^2, \tag{7.41}$$

where R_w is the 'wave Reynolds number' expressible as $c\lambda/\nu$ if c and λ denote the dimensional wave velocity and length. Thus, the effect of friction at the wave surface is to decrease the overall sheltering coefficient s for waves travelling with a fair speed compared with U_∞; however, the magnitude of s_2 is quite small since, to be consistent with other features of the physical model, R_w is large. For instance, s_2 is scarcely significant in comparison with the values of s_1 calculated by Miles (1957) for some practical examples of wind blowing over sea waves.

Incidentally, while asymptotic formulae for large z_0 are being considered, we may suitably write down an approximate expression for the shearing stress on the boundary. This is easily found by use of the formula (4.14) in conjunction with (2.12), writing $T_s = R^{-1}f''(0) = R^{-1}\{f''(0)/f(0)\}\{c-\phi(0)\}$, where the last step follows (3.5) and we have $c-\phi(0) = \epsilon c$ from (7.33). The result is

$$T_s = e^{\frac{3}{4}\pi i}\, k(U_\infty - c)^2\, (k/Rc)^{\frac{1}{2}}\, J. \tag{7.42}$$

This shows the shearing stress to be of the same order of magnitude as the 's_2' part of the pressure in phase with the wave slope. The maximum shearing stress occurs one-eighth of a wavelength downstream from the wave trough.

We return to the discussion of the sheltering coefficient, passing now to the case where z_0 is fairly small. The critical point will now lie very close to the boundary so that s_1 will be practically zero, both because U_c'' has become extremely small and because $U_c' = U'(0)$ is large. As mentioned before, this case is exemplified by a critical point in the viscous sublayer of a turbulent boundary layer, c being less than, say, $\frac{1}{3}U_\infty$. If we put $D(-z_0) = D_r + iD_i$ and use the definition $z_0 = mc/U'(0)$, equation (7.40) may be arranged to give

$$s_2 = -\left\{\frac{D_i}{(D_r - z_0)^2 + D_i^2}\right\}\left\{\frac{km(U_\infty - c)^2}{[U'(0)]^2}\right\} J^2. \tag{7.43}$$

This yields a positive value of s_2 only when $D_i < 0$, that is when $z_0 < 2{\cdot}3$, this result being the same as found in § 5 for a linear velocity profile. Values of the quantity in the first curled bracket are tabulated as a function of z_0 in table 1, having been calculated from values of D_r and D_i given by Holstein (1950). The table shows this quantity to be largest when z_0 is slightly less than 1, that is, when the critical point is at a distance slightly less than m^{-1} from the boundary. Note that (7.42) remains valid when $c \to 0$; for then $J \to I$ (equation (7.15)), and (7.42) reduces to $s_1 = 0{\cdot}644\Delta I^2$ in agreement with (7.14) in its *modified* form (I replaced by I^2) suggested a few paragraphs ago.

Remembering the significance of the sheltering coefficient with regard to energy transfer between the flow and the wave-train, it is interesting to compare the various circumstances as the wave speed increases from very small values and the critical point moves away from the wave surface. First, s is positive and s_1 is negligible; but it becomes negative when η_c increases beyond the small distance $2{\cdot}3m^{-1}$. A negative value of s implies an energy source within the waves in order to maintain c a real constant: without an energy supply the waves would presumably be damped by the action of the flow under these conditions. Eventually s becomes positive again due to the component s_1 outweighing s_2. For a logarithmic profile, Miles (1957) showed that s_1 is a maximum according to a formula like (7.39) when $k\eta_c = 0{\cdot}017$, which means when $\eta_c = 0{\cdot}0027\lambda$. When η_c increases much beyond this value, s_1 decreases very rapidly; and it is possible that s may take a small negative value due to s_2 again outweighing s_1. Finally, we note that if c is slightly negative, so that the wave moves against the flow, the theory indicates that s is still positive; but the direction of energy transfer reverses with the sign of c, so that in this case the waves tend to be damped by the flow.

z_0	G	z_0	G
0	$-0{\cdot}644$	2·5	$+0{\cdot}089$
0·5	$-1{\cdot}601$	3·0	0·183
1·0	$-1{\cdot}458$	3·5	0·155
1·5	$-0{\cdot}960$	4·0	0·088
2·0	$-0{\cdot}206$	4·5	0·038

TABLE 1. The function $G(z_0) = D_i/\{(D_r - z_0)^2 + D_i^2\}$.

For a limitingly thin boundary layer and also $R \to \infty$, it is seen that T_s, s_1 and s_2 all tend to zero (s_1 because it is $O(k\delta)$). In the limit the only stress on the wave is the in-phase pressure given by (7.34): this becomes $-ka(U_\infty - c)^2$, which is the value according to the Kelvin–Helmholtz theory.

8. Conclusion

Validity of the approximate viscous solution

Properties of the function $f(\eta)$ have been a vital consideration in many parts of this study; and, as remarked near the beginning of § 4 where the approximation to $f(\eta)$ was introduced, there is a need to examine rather carefully the assumptions forming the basis of this approximation. Several other remarks bearing on this matter have already been made; and indeed enough probably has been said here and there to cover the main points concerned. However, to round off the discussion, it seems desirable to review the assumptions in question, and in particular examine the somewhat doubtful case of turbulent boundary-layer profiles with large rates of shear near the wave surface.

We recall that $f(\eta)$ is a solution of (3.1) which, since it diminishes very rapidly with increasing η, is approximately a solution of (4.1). The accuracy of the approximate equation (4.1) requires that (i) $\alpha = k/m$ is small, and (ii) the 'linear' region of the velocity profile, over which $U = \eta U'(0)$ approximately, covers the region where the magnitude of $f(\eta)$ is still significant in comparison with its value on the boundary. The latter region is $O(m^{-1})$ in width. In § 5 where a wholly linear profile was considered, condition (i) was seen very clearly to be the only one on which the approximation to $f(\eta)$ depended. This condition would be very well satisfied for most physical applications where the Reynolds number is fairly large. It implies that the wavelength greatly exceeds the width of the friction layer, and so holds for all but extremely short waves. We may note, however, that a physical example for which α were $O(1)$ or greater could be treated very successfully by use of the linear-profile model of § 5, the exact expression (5.3) for $f(\eta)$ being kept intact. This is so because the disturbance would then be confined entirely to the region of the friction layer, so that the effects of profile curvature would very likely be negligible.

Condition (ii) need not be interpreted very stringently. For instance, if the profile starts to curve appreciably even before η gets much beyond m^{-1}, only the 'tail end' of $f(\eta)$ is affected by the curvature; and presumably the approximation based on the linear profile is still reasonably good. The assumption of (ii) will now be tested with regard to laminar and turbulent boundary-layer profiles. It will be convenient to revert to the dimensional form of notation used in the third subsection of § 7.

In a laminar boundary layer along a plane (i.e. the Blasius profile), the velocity gradient becomes 10 % less than $U'(0)$ at a distance $\eta_{10} = 1{\cdot}54XR_X^{-\frac{1}{2}}$. The size of $m\eta_{10}$ may be regarded as an indication of the validity of the assumption under consideration. We have $m = [kU'(0)/\nu]^{\frac{1}{3}}$ and $U'(0) = 0{\cdot}332U_\infty^2\,\nu^{-1}R_X^{-\frac{1}{2}}$, and hence obtain

$$m\eta_{10} = 2{\cdot}2(kX)^{\frac{1}{3}}. \tag{8.1}$$

This is quite large if the wavelength is a small fraction of X, as it must be to satisfy the assumption of approximately parallel flow. Clearly, condition (ii) is amply satisfied in applications of the theory to laminar boundary layers.

The usual estimate of the thickness of the viscous sublayer in a turbulent boundary layer is $5\nu/U_*$ (e.g. Schlichting 1955, p. 407). The velocity profile is almost exactly linear where η has less than this value, but its divergence from the straight line $\eta U'(0)$ is not very marked until η takes considerably larger values (this is best seen by plotting the numerous available experimental measurements of $U(\eta)$ against a linear scale of η, instead of the usual logarithmic scale). For the present purpose, the value $\eta = 10\nu/U_*$ is still a fairly conservative estimate of the limit within which the viscous solution should become small. A suitable criterion as required is therefore that $10m\nu/U_*$ should be at least about unity, and of course preferably larger. Since $U'(0) = U_*^2/\nu = c_f U_\infty^2/\nu$, this criterion can be expressed

$$\left.\begin{aligned} &10(k\nu/U_*)^{\frac{1}{3}} > \sim 1,\\ \text{or}\quad &18{\cdot}5c_f^{-\frac{1}{6}}(\lambda U_\infty/\nu)^{-\frac{1}{3}} > \sim 1.\end{aligned}\right\} \tag{8.2}$$

With Motzfeld's experimental values $U_\infty\lambda/\nu = 330{,}000$ and $c_f = 0{\cdot}00173$ quoted in § 7, the left-hand side of (8.2) is 0·77. Thus, the approximation is probably still reasonably accurate, although this would appear to be a marginal case. However, the value of U_* in the experiments (about 80 cm/sec) was somewhat exceptionally large. A value $U_* = 10$ cm/sec, for instance, is typical of a light to moderate wind over water, the flow being 'aerodynamically smooth'. With this value of U_* and the typical value for air $\nu = 0{\cdot}17\,\text{cm}^2/\text{sec}$, the left-hand side of (8.2) is unity for $\lambda = 107$ cm. It could therefore be expected that the approximate viscous solution would be quite reliable for waves less than about a metre in length.

A generalization by Fourier's theorem

We reconsider (5.6) and (5.9), which express the normal and tangential stresses p_s and τ_s on a solid or slowly moving wave bounding a uniform shear flow. The simple yet somewhat unusual form of these results makes it attractive to generalize them, by Fourier's theorem, so as to apply when the boundary is an arbitrary perturbation from a plane. This will provide a particular clear illustration of the idea, which has been frequently mentioned, that a type of sheltering action occurs even according to linearized theory when the stresses bear certain phase relations to the wave. Only the first-order terms in (5.6) and (5.9) will be retained, since the terms having α as a factor can be regarded as of secondary importance.

It then follows that if the deformation in the boundary is non-periodic and given by

$$y = \zeta(x) = \frac{1}{\pi}\int_0^\infty \{g_1(k)\cos kx + g_2(k)\sin kx\}\,dk, \tag{8.3}$$

where, according to Fourier's theorem,

$$g_1(k) = \int_{-\infty}^{\infty} \zeta(x)\cos kx\,dx, \quad g_2(k) = \int_{-\infty}^{\infty} \zeta(x)\sin kx\,dx, \tag{8.4}$$

then the stresses are

$$p_s = -\frac{A}{\pi}\int_0^\infty \{g_1(k)\cos(kx - \tfrac{1}{6}\pi) + g_2(k)\sin(kx - \tfrac{1}{6}\pi)\}\,k^{-\frac{1}{3}}\,dk, \tag{8.5}$$

and
$$\tau_s = \frac{B}{\pi}\int_0^\infty \{g_1(k)\cos(kx + \tfrac{1}{6}\pi) + g_2(k)\sin(kx + \tfrac{1}{6}\pi)\}\,k^{\frac{1}{3}}\,dk, \tag{8.6}$$

where $A = 0{\cdot}776R^{-\frac{1}{3}}G^{\frac{4}{3}}$ and $B = 1{\cdot}065R^{-\frac{2}{3}}G^{\frac{5}{3}}$.

Consider, for example, the case where the boundary profile is the single-humped curve

$$y = \frac{ab^2}{x^2 + b^2}. \tag{8.7}$$

One finds without difficulty that $g_1(k) = \pi ab\,e^{-kb}$ and $g_2(k) = 0$. Hence, (8.5) and (8.6) give

$$\begin{aligned} p_s &= -Aab\int_0^\infty e^{-kb}\cos(kx - \tfrac{1}{6}\pi)\,k^{-\frac{1}{3}}\,dk \\ &= -Aab\Gamma(\tfrac{2}{3})\,(x^2+b^2)^{-\frac{1}{3}}\cos\left(\tfrac{2}{3}\tan^{-1}\frac{x}{b} - \tfrac{1}{6}\pi\right); \end{aligned} \tag{8.8}$$

$$\begin{aligned} \tau_s &= Bab\int_0^\infty e^{-kb}\cos(kx + \tfrac{1}{6}\pi)\,k^{\frac{1}{3}}\,dk \\ &= \tfrac{1}{3}Bab\Gamma(\tfrac{1}{3})\,(x^2+b^2)^{-\frac{2}{3}}\cos\left(\tfrac{4}{3}\tan^{-1}\frac{x}{b} + \tfrac{1}{6}\pi\right). \end{aligned} \tag{8.9}$$

To illustrate how the stresses are distributed over the boundary profile, figure 5 shows graphs of the functions $\frac{1}{2}(x^2+1)^{-1}$, $(x^2+1)^{-\frac{1}{3}}\cos(\frac{2}{3}\tan^{-1}x - \frac{1}{6}\pi)$, and $(x^2+1)^{-\frac{2}{3}}\cos(\frac{4}{3}\tan^{-1}x + \frac{1}{6}\pi)$. The interpretation to be given the three curves is perfectly clear in the light of (8.7), (8.8) and (8.9). The figure shows the pressure to be everywhere negative (note that $-p_s$ is represented), being greatest in magnitude slightly on the leeward side of the boundary hump. The shearing stress changes sign and becomes negative farther on the leeward side, indicating a tendency towards a reversal of the flow some way downstream. These features have the same general character as if the flow separated on the downstream side, thus causing sheltering in the usually understood sense. Of course, the present flow does not separate: the point being emphasized is that an infinitesimal disturbance can produce effects similar to those due to separation.

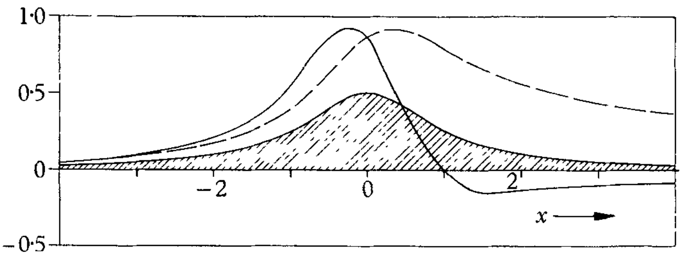

FIGURE 5. Graphs of the stresses on a solid hump due to uniform shearing flow in the x-direction. The curve shaded underneath represents the boundary profile. The other full-line curve represents the shearing stress; and the dashed line represents the (negative) pressure. The vertical scale is arbitrary.

Applications of formulae for the surface stresses

Aside from the intrinsic interest they may have, the various formulae that have been developed for the stresses on the wavy boundary offer the possibility of usefulness in problems concerning wave generation by the action of a flow. For instance, they might be applied to the study of two-dimensional flutter waves on a membrane. The present theory has been restricted to real values of the wave velocity c; but this is relevant to conditions of 'neutral stability' under which a simple-wave disturbance in a system is propagated unchanged, the factors tending respectively to damp or amplify the wave being exactly balanced. The determination of neutral conditions is usually the first aim of stability investigations.

Although the principle of the method whereby the stress formulae would be applied is perhaps fairly obvious, it seems worth outlining explicitly. Suppose wave formation is expected to occur in a certain deformable body whose interface with the fluid is a plane; and suppose the physical properties of the undisturbed flow are specified, including the velocity profile. The method assumes the interface to be perturbed by a simple wave, of infinitesimal amplitude a and *arbitrary* wave-number k and velocity c, which by appeal to the principle of Fourier synthesis covers every physically possible form of infinitesimal disturbance. Now, the present formulae provide estimates of the periodic stresses, proportional to a and depending on k and c, which the flow exerts against the wave. Hence, complete dynamical boundary conditions can be formulated which, together with a kinematical condition expressible in terms of k and c, determine the motion of the body. The action of the flow is thus regarded as an influence analogous to, say, surface tension—which for a given deformation produces an easily calculable normal force. The solution of the separate dynamical problem for the body would in general lead to a relation between k and c, with various physical properties such as viscosity or stiffness of the body involved as parameters (in effect, the connexion between k and c would emerge as a condition of 'self-consistency' of the dynamical system: if c is allowed to be complex, waves with any k and c_r are of course possible; but for a given k only certain discrete values of c_r, if any, are consistent with $c_i = 0$). This relation would define the class of neutral waves; and from the result there would probably be little difficulty in deciding the combinations of physical parameters for which instability may occur (cf. Feldman (1957), where stability conditions involving a large number of parameters are discussed).

When the second medium is another fluid, strictly speaking another property of the wave interface enters the stress formulae, namely β (equation (2.9)) which relates to the slipping of particles in the surface relative to their positions on the undisturbed plane. However, several reasons have been given why β may be negligible; and it would seem that the effect of β would be quite unimportant in most problems concerning a gas-liquid interface, i.e. the stresses are the same as on a similar wave moving over a flexible solid. Also, one may obviously expect the stresses to become approximately the same as on a fixed solid when c is sufficiently small; but a more precise criterion for this has been seen to be that $mc/U'(0)$ should be $O(1)$ or less.

An alternative approach to problems of wave formation in a two-phase system is to set out mathematics at once embracing the whole model, yet likely to be of such complexity as to lose sight of physical reasoning. In contrast, the 'divided attack' suggested here has the advantage that a helpful stocktaking from a physical viewpoint can be made at the intermediate stage. For instance, it can be seen whether or not the surface stresses supply energy to the wave, which action is of course necessary for the survival of a neutral wave when the second medium is dissipative. Further, the relative importance of the normal and shearing stresses can be well understood with reference to energy considerations; and a qualitative assessment of the conditions most likely to promote waves may be possible even without a detailed study of the mechanics of the second medium. Again, the normal stress component in phase with the wave can be neatly interpreted as an accession (if the stress is positive) or decrement to the inertia of the second medium (we recall that Kelvin–Helmholtz instability is due solely to the inertial effects of the upper fluid).

We emphasize, however, that energy arguments serve best to provide only some additional physical insight into problems of wave formation, and are preferably not to be relied upon alone for the deduction of stability conditions. Energy methods are well known to be unreliable in the classical problems of hydrodynamic stability (Lin 1955, § 4.5; Schlichting 1955, p. 313). Solution of the linearized dynamical equations is a better course, being concerned with quantities of the first rather than second order in the wave amplitude. If the question is wave generation by the action of flow on a mobile boundary, the way is open to the latter approach when, as provided here, the interfacial stresses are completely known.

REFERENCES

ERDÉLYI, A. (ED.) 1954 *Tables of Integral Transforms*, Vol. 1. New York: McGraw-Hill.

FELDMAN, S. 1957 *J. Fluid Mech.* **2**, 343.

GOLDSTEIN, S. (ED.) 1938 *Modern Developments in Fluid Dynamics*. Oxford University Press.

HOLSTEIN, H. 1950 *Z. angew. Math. Mech.* **30**, 25.

JEFFREYS, H. 1924 *Proc. Roy. Soc.* A, **108**, 189.

JEFFREYS, H. 1925 *Proc. Roy. Soc.* A, **110**, 241.

KNUTH, E. H. 1954 *Jet Propulsion*, **24**, 359.

LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

LIGHTHILL, M. J. 1953 *Proc. Roy. Soc.* A, **217**, 478.

LIGHTHILL, M. J. 1957 *J. Fluid Mech.* **3**, 113.

LIN, C. C. 1945 *Quart. Appl. Math.* **3**, Part 1, p. 117; Part 2, p. 218; Part 3, p. 277.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

LOCK, R. C. 1954 *Proc. Camb. Phil. Soc.* **50**, 105.

MILES, J. W. 1957 *J. Fluid Mech.* **3**, 185.

MOTZFELD, H. 1937 *Z. angew. Math. Mech.* **17**, 193.

PHILLIPS, O. M. 1957 *J. Fluid Mech.* **2**, 417.

QUICK, A. W. & SCHRÖDER, K. 1944 Wartime report summarized by W. Mangler in *Ministry of Aircraft Production Rep. and Trans.*, no. 1003, §2 (1948).

SCHLICHTING, H. 1955 *Boundary Layer Theory*. London: Pergamon.

STANTON, T. E., MARSHALL, D. & HOUGHTON, R. 1932 *Proc. Roy. Soc.* A, **137**, 283.

TOLLMIEN, W. 1929 *Nachr. Ges. Wiss. Göttingen*, **1**, 21; English translation in *Nat. Adv. Comm. Aero., Wash., Mem.* no. 609.

URSELL, F. 1956 Wave generation by wind, article in *Surveys in Mechanics* (Ed. Batchelor and Davies). Cambridge University Press.

WUEST, W. 1949 *Z. angew. Math. Mech.* **29**, 239.

ZONDEK, B. & THOMAS, L. H. 1953 *Phys. Rev.* **90**, 738.